ON THE RATE OF CONVERGENCE IN DUFFUSION APPROXIMATION OF JUMP MARKOV PROCESSES

By

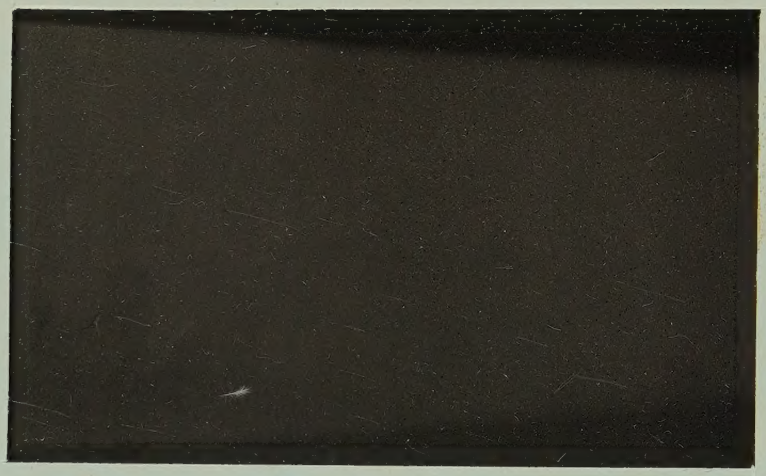
Sven Erick Alm

U.U.D.M. Report No. 1978:8

UPPSALA UNIVERSITY

DEPARTMENT OF MATHEMATICS

1978



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Mathematical Statistics, Department of Mathematics, Uppsala University.
In English.

Summary:

Families of d-dimensional jump Markov processes $Y^{(\varepsilon)}(t)$, with jump intensities $\varepsilon^{-1}\lambda^{(\varepsilon)}(y)$ and jump measure $\mu^{(\varepsilon)}(y)$, are studied. Under approximate conditions Kurtz has shown that, as $\varepsilon \to 0$, $\xi^{(\varepsilon)}(t) = \varepsilon(Y)^{(\varepsilon)}(t) \xrightarrow{P} \bar{\xi}(t)$, where $\bar{\xi}(t)$ is a deterministic function, and $X^{(\varepsilon)}(t) = \varepsilon^{-1/2}(\xi^{(\varepsilon)}(t) - \bar{\xi}(t)) \xrightarrow{W} X(t)$, where $X(t)$ is a diffusion process.

An operator technique is used to prove that the rate of convergence of the distribution function of $X^{(\varepsilon)}(t)$ to that of $X(t)$ is at most $K(t)\varepsilon^{1/2}$, eliminating a logarithm that appears in previous results. Explicit expressions for $K(t)$ are obtained and in particular it is shown that, under a certain stability condition, $K(t)$ is decreasing.

Examples from epidemiology and chemistry are given.

Key words and phrases: Markov process, diffusion approximation, rate of convergence, operator method.

Doctoral dissertation to be publicly examined in Room 309, Department of Mathematics, Uppsala University, on December 2, 1978, at 10 a.m., for the degree of Doctor of Philosophy (according to Royal Proclamation No. 327, 1969).

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Report No. 8, November 1978

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The theorems are stated and proved for $d=1$ in §5 and for $d>1$ in §6. The generalization to the latter case involves some non-trivial complications which is the motivation for separating the proofs. Some lemmas which are used in both cases are placed in §4.

Applications to epidemic and chemical processes are given in §7.

Finally, the proof of Lemma 4.1 is placed in Appendix A as it contains mainly well-known results and the proof of Lemma 4.2, which verifies the forward and backward equations, is found in Appendix B.

Acknowledgement. My teacher, Dr. Anders Martin-Löf, suggested the topic of this paper and I wish to express my deep gratitude to him for very stimulating discussions and his most valuable advice and support. I also wish to thank Prof. Carl-Gustav Esseen for his kind interest in my work.

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§2. Notations and previous results

Let the operator corresponding to $Y^{(\varepsilon)}$ be $\tilde{G}^{(\varepsilon)}_{s,t}$, that is

$$(\tilde{G}^{(\varepsilon)}_{s,t}\varphi)(y) = E(\varphi(Y^{(\varepsilon)}(t)) \mid Y^{(\varepsilon)}(s) = y) \quad \text{for} \quad \varphi \in C(R^d),$$

the class of bounded continuous functions on R^d. It is well known that if

$$\sup_{\varepsilon}\sup_{y} \beta^{(\varepsilon)}(y, R^d) = \sup_{\varepsilon}\sup_{y} \lambda^{(\varepsilon)}(y) < \infty \tag{2.1}$$

then $\tilde{G}^{(\varepsilon)}_{s,t}$ satisfies the backward equation

$$-\frac{\partial}{\partial s}\tilde{G}^{(\varepsilon)}_{s,t} = \tilde{A}^{(\varepsilon)}\tilde{G}^{(\varepsilon)}_{s,t} \quad \text{with} \quad G^{(\varepsilon)}_{t,t} = I,$$

where I is the identity operator and

$$(\tilde{A}^{(\varepsilon)}\varphi)(y) = \varepsilon^{-1}\int (\varphi(y+z) - \varphi(y))\beta^{(\varepsilon)}(y,dz).$$

For a survey of Markov processes see [7].

$\xi^{(\varepsilon)}(t) = \varepsilon \cdot Y^{(\varepsilon)}(t)$ is a new sequence of jump Markov processes with operators $\hat{G}^{(\varepsilon)}_{s,t}$ defined on $C(R^d)$ and satisfying

$$-\frac{\partial}{\partial s}\hat{G}^{(\varepsilon)}_{s,t} = \hat{A}^{(\varepsilon)}\hat{G}^{(\varepsilon)}_{s,t} \quad \text{with} \quad \hat{G}^{(\varepsilon)}_{t,t} = I,$$

where

$$(\hat{A}^{(\varepsilon)}\varphi)(\xi) = \varepsilon^{-1}\int (\varphi(\xi+\varepsilon z) - \varphi(\xi))\ \alpha^{(\varepsilon)}(\xi,dz)$$

and

$$\alpha^{(\varepsilon)}(\xi,A) = \beta^{(\varepsilon)}(\varepsilon^{-1}\xi,A).$$

Let

$$b^{(\varepsilon)}(\xi) = \int z\alpha^{(\varepsilon)}(\xi, dz),$$

$$C^{(\varepsilon)}(\xi) = \int zz^T \alpha^{(\varepsilon)}(\xi, dz),$$

$$B^{(\varepsilon)}(\xi) = \frac{db^{(\varepsilon)}(\xi)}{d\xi} = \left(\frac{\partial}{\partial \xi_j} b_i^{(\varepsilon)}(\xi)\right)_{d\times d}.$$

The notations $b(\xi)$, $C(\xi)$ and $B(\xi)$ are used when $\alpha^{(\varepsilon)} = \alpha$ which is independent of ε. For a vector $x = (x_1,\ldots,x_d)$ let $|x| = (\sum_i x_i^2)^{1/2}$ and for a matrix $A = (a_{ij})$ let $|A| = (\sum_{i,j} x_{ij}^2)^{1/2}$.

Let $\bar{\xi}(t)$ be a solution to the differential equation

$$\xi'(t) = b(\xi(t)) \quad \text{with} \quad \xi(0) = \xi_0. \tag{2.2}$$

In [10] Kurtz proves the following theorem:

Theorem 2.1. Suppose that condition (2.1) is satisfied, that $\sup_{\varepsilon}\sup_{\xi}\int |z|\,\alpha^{(\varepsilon)}(\xi, dz) < \infty$ and that there exists a function $\delta(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ such that

$$\limsup_{\varepsilon\to 0}\sup_{\xi}\int_{|z|>\delta(\varepsilon)/\varepsilon} |z|\,\alpha^{(\varepsilon)}(\xi, dz) = 0.$$

Suppose further the existence of a function b which is Lipschitz continuous and such that

$$\limsup_{\varepsilon\to 0}\sup_{\xi} |b^{(\varepsilon)}(\xi) - b(\xi)| = 0.$$

Let $\bar{\xi}(t)$ be a solution of (2.2). If $\lim_{\varepsilon\to 0}\xi^{(\varepsilon)}(0) = \xi_0$ then for every $t \geq$

$$\sup_{s\leq t} |\xi^{(\varepsilon)}(s) - \bar{\xi}(s)| \xrightarrow{P} 0.$$

Remark 2.1: If the initial values $\xi^{(\varepsilon)}(0)$ are allowed to be random, it suffices that $\xi^{(\varepsilon)}(0) \xrightarrow{P} \xi_0$.

The processes $X^{(\varepsilon)}(t) = \varepsilon^{-1/2}(\xi^{(\varepsilon)}(t) - \bar{\xi}(t))$, with $\bar{\xi}(t)$ a solution of (2.2), are a new sequence of Markov processes but they are no longer pure jump processes nor temporally homogeneous because of $\bar{\xi}(t)$. Let the operator of $X^{(\varepsilon)}(t)$ given $X^{(\varepsilon)}(s)$ be denoted by $G^{(\varepsilon)}_{s,t}$ and let

$$G^{(\varepsilon)}_{s,t}(\eta|x) = P(X^{(\varepsilon)}(t) \leq \eta \mid X^{(\varepsilon)}(s) = x),$$

where $x \leq \eta$ is understood as $x_1 \leq \eta_1, \ldots, x_d \leq \eta_d$. For simplicity, let $B(t) = B(\bar{\xi}(t))$ and $C(t) = C(\bar{\xi}(t))$.

In [11] Kurtz proves the following theorem in a slightly stronger formulation.

Theorem 2.2. Suppose that condition (2.1) is satisfied, that $\lim_{\varepsilon\to 0}\sup_{\xi}\varepsilon^{-1/2}|b^{(\varepsilon)}(\xi) - b(\xi)| = 0$ where $b(\xi)$ is bounded, Lipschitz continuous and has uniformly continuous first partial derivatives $B(\xi)$ and that $C^{(\varepsilon)}(\xi)$ converges uniformly to $C(\xi)$ which is bounded and uniformly continuous.

If there is a function $\delta(\varepsilon)$, $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\limsup_{\varepsilon \to 0} \sup_{\xi} \int_{|z| > \delta(\varepsilon)\varepsilon^{-1/2}} |z|^2 \alpha^{(\varepsilon)}(\xi, dz) = 0,$$

then $\lim_{\varepsilon \to 0} \varepsilon^{-1/2}(\xi^{(\varepsilon)}(0) - \xi_0) = x_0$ implies that $X^{(\varepsilon)} \xrightarrow{w} X$ in $D[0,T]$, $T \geq 0$, where $X(t)$ is a diffusion process with $X(0) = x_0$ and characteristic function $\psi(t,\theta)$ satisfying

$$\frac{\partial}{\partial t} \psi(t,\theta) = -\frac{1}{2} \cdot \theta^T C(t)\theta + \theta^T \cdot B(t) \cdot \frac{d}{d\theta} \psi(t,\theta).$$

Remark 2.2. By $\frac{d\psi}{d\theta}$ is meant the vector of partial derivatives.

Remark 2.3. It is sufficient that $X^{(\varepsilon)}(0) \xrightarrow{P} X(0)$, where $X(0)$ may be stochastic.

The diffusion $X(t)$ of Theorem 2.2 can be expressed as an Itô integral

$$X(t) = \mu(0,t)X(0) + \int_0^t \mu(s,t)\, C^{1/2}(s) dW(s), \tag{2.3}$$

where $W(s)$ is a d-dimensional Wiener process and the matrix $\mu(s,t)$ is the solution of

$$\frac{\partial}{\partial t} \mu(s,t) = B(t)\mu(s,t) \quad \text{with} \quad \mu(s,s) = I. \tag{2.4}$$

From (2.3) it is easily seen that $X(t)$ is a Gauss-Markov process which given $X(s) = x$ has expectation $\mu(s,t)\,x$ and covariance matrix

$$C(s,t) = \int_s^t \mu(r,t)C(r)\mu^T(r,t)dr. \tag{2.5}$$

Useful properties of $\mu(s,t)$ and $C(s,t)$, defined by (2.4) and (2.5), are collected in Lemma 4.1, which is proved in Appendix A.

Let $X(t)$ given $X(s)$ have operator $G_{s,t}$ and distribution function $G_{s,t}(\eta|x) = P(X(t) \leq \eta \mid X(s) = x)$.

A third way of characterizing $X(t)$ is through the easily verified backward equation

$$-\frac{\partial}{\partial s} G_{s,t}\varphi = B_s\, G_{s,t}\varphi, \tag{2.6}$$

where $(B_s\varphi)(x) = (B(s)x) \cdot \varphi'(x) + \frac{1}{2}$ trace $(C(s)\varphi''(x))$, which holds for

$\varphi \in C^2(R^d)$, the class of bounded continuous functions on R^d with all second order partial derivatives bounded and continuous. By $\varphi'(x) = \frac{d}{dx}\varphi(x)$ is understood the vector of first order partial derivatives and by $\varphi''(x) = \frac{d^2}{dx^2}\varphi(x)$ the matrix of second order partial derivatives.

Let $|S|$ denote the number of elements in the set S.

For a matrix A, $(d \times d)$, the eigenvalues will be denoted $\lambda_1(A) \leq \lambda_2(A) \leq \ldots \leq \lambda_d(A)$ if they are all real.

Introduce $D^{(\varepsilon)}_{s,t} = G^{(\varepsilon)}_{s,t} - G_{s,t}$ and $D^{(\varepsilon)}_{s,t}(\eta|x) = G^{(\varepsilon)}_{s,t}(\eta|x) - G_{s,t}(\eta|x)$.

§3. General assumptions

The following assumptions will be made frequently in the sequel.

$$\sup_{\varepsilon}\sup_{y}\lambda^{(\varepsilon)}(y) < \infty. \tag{3.1}$$

With $\bar{a} = \sup_t (-\lambda_1((B(t)+B^T(t))/2))$ and $\underline{a} = \inf_t (-\lambda_d((B(t)+B^T(t))/2))$

$$-\infty < \underline{a} \le \bar{a} < \infty. \tag{3.2}$$

For $\bar{c} = \sup_t \lambda_d(C(t))$ and $\underline{c} = \inf_t \lambda_1(C(t))$

$$0 < \underline{c} \le \bar{c} < \infty. \tag{3.3}$$

$$\sup_t |B(t)| < \infty. \tag{3.4}$$

For all x,y: $|b(y) - b(x) - B(x)\cdot(y-x)| \le K|y-x|^2$. (3.5)

For all x,y: $|C(y) - C(x)| \le K|y-x|$. (3.6)

$$\sup_x |C(x)| < \infty. \tag{3.7}$$

Remark 3.1. Observe that (3.7) implies

$$\sup_x \int |z|^2 \alpha(x,dz) < \infty, \tag{3.8}$$

which together with (3.1) implies

$$\sup_x \int |z| \alpha(x,dz) < \infty \tag{3.9}$$

and

$$\sup_x |b(x)| < \infty. \tag{3.10}$$

Further (3.4) implies (3.2) and (3.7) implies the right part of (3.3).

§4. Preparatory lemmas

In this chapter it will be assumed throughout that the assumptions of §3 are satisfied. It will also be assumed that $\alpha^{(\varepsilon)}(\xi,A) = \alpha(\xi,A)$ which is independent of ε.

The first lemma, on properties of the limit distribution, partly contains well-known facts but is stated here for convenience. The proof is found in Appendix A.

Lemma 4.1.

(i) $\mu(s,t) \cdot \mu(r,s) = \mu(r,t)$.

(ii) $d^{1/2}\, e^{-\bar{a}(t-s)} \le |\mu(s,t)| \le d^{1/2}\, e^{-\underline{a}(t-s)}$,

$d^{1/2}\, e^{\underline{a}(t-s)} \le |\mu^{-1}(s,t)| \le d^{1/2}\, e^{\bar{a}(t-s)}$.

(iii) $|\mu(s,t) - I| \le K(1 - e^{-\underline{a}(t-s)})/\underline{a}$.

(iv) $e^{-K(t-s)} \le \det \mu(s,t) \le e^{K(t-s)}$.

There exists a constant $\delta_0 > 0$ such that if $t - s \le \delta_0$ then $\det \mu_\Omega(s,t) > 1/2$, where the matrix μ_Ω, of elements in μ, is obtained by deleting those rows and columns in μ not corresponding to indices in the set Ω.

(v) $K\cdot(1 - e^{-2\bar{a}(t-s)})/\bar{a} \le |C(s,t)| \le K(1 - e^{-2\underline{a}(t-s)})/\underline{a}$.

(vi) $K\cdot((1 - e^{-2\bar{a}(t-s)})/\bar{a})^{1/2} \le |C^{1/2}(s,t)| \le K((1 - e^{-2\underline{a}(t-s)})/\underline{a})^{1/2}$.

(vii) $K\cdot((1 - e^{-2\underline{a}(t-s)})/\underline{a})^{-1/2} \le |C^{-1/2}(s,t)| \le K((1 - e^{-2\bar{a}(t-s)})/\bar{a})^{-1/2}$

(viii) $K((e^{2\bar{a}(t-s)} - 1)/\bar{a})^{-1/2} \le |C^{-1/2}(s,t)\cdot\mu(s,t)| \le K((e^{2\underline{a}(t-s)} - 1)/\underline{a})^{-1/2}$

(ix) $\min_i C_{ii}(s,t) \ge K(1 - e^{-2\bar{a}(t-s)})/\bar{a}$.

Remark 4.1. Here and in the sequel K is used to denote different constants only depending on the measure α not on time or ε.

Remark 4.2. The lemma is valid for all values of $\underline{a}$ and $\bar{a}$ with the natural interpretation of $(1 - e^{-at})/a$ as t when $a = 0$.

The following lemma gives the backward and forward equations of the operator $G^{(\varepsilon)}_{s,t}$ connected with $X^{(\varepsilon)}(t)|X^{(\varepsilon)}(s)$. The proof is found in Appendix B.

Lemma 4.2. Suppose that $\alpha'_k(x,y) = \frac{\partial}{\partial x_k}\alpha(x,y)$ exists for $k = 1,2,\ldots,d$ and that

$$\max_k \sup_x \int |\alpha'_k(x,dz)| < \infty . \tag{4.1}$$

Then, for $\varphi \in C^1(R^d)$, $G^{(\varepsilon)}_{s,t}\varphi$ satisfies the backward equation

$$-\frac{\partial G^{(\varepsilon)}_{s,t}\varphi}{\partial s} = A^{(\varepsilon)}_s G^{(\varepsilon)}_{s,t}\varphi \quad \text{with} \quad G^{(\varepsilon)}_{t,t} = I$$

and the forward equation

$$\frac{\partial G^{(\varepsilon)}_{s,t}\varphi}{\partial t} = G^{(\varepsilon)}_{s,t}A^{(\varepsilon)}_t\varphi \quad \text{with} \quad G^{(\varepsilon)}_{s,s} = I ,$$

where $(A^{(\varepsilon)}_s\varphi)(x) = \varepsilon^{-1}\int (\varphi(x+\varepsilon^{1/2}z) - \varphi(x))\,\alpha(\bar{\xi}(s)+\varepsilon^{1/2}x,dz) -$
$- \varepsilon^{-1/2}\,\varphi'(x)\cdot b(\bar{\xi}(s))$.

Define $\varphi_\eta(x) = \begin{cases} 1 & x \le \eta \\ 0 & \text{otherwise.} \end{cases}$

The object is to estimate

$$D^{(\varepsilon)}_{s,t}(\eta|x) = G^{(\varepsilon)}_{s,t}(\eta|x) - G_{s,t}(\eta|x) = (G^{(\varepsilon)}_{s,t}\varphi_\eta)(x) - (G_{s,t}\varphi_\eta)(x) = (D^{(\varepsilon)}_{s,t}\varphi_\eta)(x) .$$

A practical working formula for $D^{(\varepsilon)}_{s,t}$ is given in

Lemma 4.3. Under Condition (4.1)

$$D^{(\varepsilon)}_{s,t} = \int_s^t G_{s,r}(A^{(\varepsilon)}_r - B_r)G_{r,t}dr + \int_s^t D^{(\varepsilon)}_{s,r}(A^{(\varepsilon)}_r - B_r)G_{r,t}dr .$$

Proof. From (2.6) and Lemma 4.2 follows

$$\frac{d}{dr}G^{(\varepsilon)}_{s,r}G_{r,t} = G^{(\varepsilon)}_{s,r}A^{(\varepsilon)}_r G_{r,t} - G^{(\varepsilon)}_{s,r}B_r G_{r,t} = G^{(\varepsilon)}_{s,r}(A^{(\varepsilon)}_r - B_r)G_{r,t}$$

so that

$$D^{(\varepsilon)}_{s,t} = G^{(\varepsilon)}_{s,t} - G_{s,t} = \int_0^t \frac{d}{dr}G^{(\varepsilon)}_{s,r}G_{r,t}dr = \int_s^t G^{(\varepsilon)}_{s,r}(A^{(\varepsilon)}_r - B_r)G_{r,t}dr =$$

$$= \int_s^t D^{(\varepsilon)}_{s,r}(A^{(\varepsilon)}_r - B_r)G_{r,t}dr + \int_s^t G_{s,r}(A^{(\varepsilon)}_r - B_r)G_{r,t}dr .$$

The next step is to estimate the second order moments of $G_{0,s}$ and $G^{(\varepsilon)}_{0,s}$. Introduce

$$R(x,y) = \begin{cases} \sum_k (b_k(x) - b_k(y))(x_k - y_k)/|x-y|^2 & \text{if } x \neq y \\ \lambda_d((B(y)+B^T(y))/2) & \text{if } x = y \end{cases}$$

and

$$\alpha = \sup_s \sup_x R(x, \bar{\xi}(s)) . \tag{4.2}$$

$R(x,y)$ can also be expressed as

$$R(x,y) = \int_0^1 dt \sum_k \sum_\ell (x_k - y_k)(x_\ell - y_\ell)\cdot B_{k\ell}(y+t(x-y))/|x-y|^2 \leq$$

$$\leq \int_0^1 dt\, \lambda_d((B(y+t(x-y)) + B^T(y+t(x-y))/2),$$

which motivates the definition for $x=y$ and also gives

$\alpha \geq \sup_s R(\bar{\xi}(s), \bar{\xi}(s)) = \sup_s \lambda_d((B(s)+B^T(s))/2) = -\underline{a}$. Note also that

$$\alpha \leq \sup_\xi \lambda_d((B(\xi)+B^T(\xi))/2) . \tag{4.3}$$

Lemma 4.4.

(i) $$E(|X(s)|^2 | X(0)=x) \leq K(s,-\underline{a})(1+|x|^2) \leq K(s,\alpha)(1+|x|^2) ,$$

where

$$K(s,a) = \begin{cases} K & \text{if } a < 0 \\ K(1+s) & \text{if } a = 0 \\ K\, e^{2as} & \text{if } a > 0 . \end{cases}$$

(ii) $$E(|X^{(\varepsilon)}(s)|^2 | X^{(\varepsilon)}(0)=x) \leq K(s,\alpha)(1+|x|^2)$$

if (4.1) is satisfied.

Proof. (i) $(X(s)|X(0)=x) \overset{d}{=} N(\mu(0,s)x, C(0,s))$ implies

$$E(|X(s)|^2 | X(0)=x) = E(\sum_{i=1}^d X_i^2(s) | X(0)=x) =$$

$$= \sum_{i=1}^d (C_{ii}(0,s) + (\mu(0,s)x)_i^2) \leq \sqrt{d}\, |C(0,s)| + |\mu(0,s)|^2 \cdot |x|^2 \leq$$

$$\leq K(1-e^{-2\underline{a}s})/\underline{a} + de^{-2\underline{a}s}\, |x|^2 .$$

For the proof of (ii) a lemma, similar to Lemma 2.9 of [11], is needed

Lemma 4.5. If (4.1) is satisfied then

(i) For every $\varphi \in C^1(R^d)$

$$Z(t) = \varphi(X^{(\varepsilon)}(t)) - \varphi(X^{(\varepsilon)}(0)) - \int_0^t (A_s^{(\varepsilon)}\varphi)(X^{(\varepsilon)}(s))ds$$

is a martingale with respect to $\mathcal{F}_t = \sigma(X^{(\varepsilon)}(s),\ s \le t)$.

(ii) If $E|X^{(\varepsilon)}(0)| < \infty$ then $Z(t)$ is a martingale also for $\varphi(x) = (1 + |x|^2)^{1/2}$.

(iii) If $E|X^{(\varepsilon)}(0)|^2 < \infty$ then $Z(t)$ is a martingale also for $\varphi(x) = |x|^2$.

Proof. (i) $Z(\tau) - Z(t) = \varphi(X^{(\varepsilon)}(\tau)) - \varphi(X^{(\varepsilon)}(t)) - \int_t^\tau (A_s^{(\varepsilon)}\varphi)(X^{(\varepsilon)}(s))ds$ when $\tau \ge t$.

$\sup_x |\varphi(x)| \le K$ and $\sup_x |\varphi'(x)| \le K$ as $\varphi \in C^1(R^d)$ so that

$$|(A_s^{(\varepsilon)}\varphi)(x)| \le \varepsilon^{-1} \int |\varphi(x+\varepsilon^{1/2}z) - \varphi(x)|\ \alpha(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) +$$

$$+ \varepsilon^{-1/2}|\varphi'(x)|\,|b(\bar{\xi}(s))| \le K\cdot\varepsilon^{-1} + K\varepsilon^{-1/2} < \infty.$$

Thus, $E|Z(t)| \le 2K + t(K\varepsilon^{-1} + K\varepsilon^{-1/2}) < \infty$ for each $t < \infty$. As $X^{(\varepsilon)}(t)$ is a Markov process, the condition $E(Z(\tau) - Z(t)\,|\mathcal{F}_t) = 0$ is equivalent to

$$(G_{t,\tau}^{(\varepsilon)}\varphi)(X^{(\varepsilon)}(t)) - \varphi(X^{(\varepsilon)}(t)) = \int_t^\tau (G_{t,s}^{(\varepsilon)}A_s^{(\varepsilon)}\varphi)(X^{(\varepsilon)}(t))ds \quad \text{a.s.},$$

which is just the integrated form of the forward equation, which by Lemma 4.2 holds under (4.1).

(ii) $$\varphi(x) = (1 + |x|^2)^{1/2} = (1 + \sum_{k=1}^d x_k^2)^{1/2}.$$

$$\varphi_k'(x) = x_k/(1 + |x|^2)^{1/2}.$$

Choose a sequence

$$\varphi^{(n)}(x) = (1 + |x|^2)^{1/2}/(1 + |x|^2/n^2)^{1/2} \nearrow \varphi(x).$$

Further, $0 \le \varphi^{(n)}(x) \le n$ and

$$\varphi_k^{(n)\prime}(x) = n(n^2 - 1)x_k/((n^2 + |x|^2)^{3/2}(1 + |x|^2)^{1/2}),$$

so that $\lim_{n\to\infty} \varphi_k^{(n)\prime}(x) = \varphi_k'(x)$ and $|\varphi^{(n)\prime}(x)|^2 \le |x|^2/(1+|x|^2) \le 1$,
which gives

$$|\varphi^{(n)}(x+v) - \varphi^{(n)}(x)| \le |v| . \tag{4.4}$$

Thus $\varphi^{(n)} \in C^1(R^d)$ for all $n < \infty$ so that by (i)

$$Z^{(n)}(t) = \varphi^{(n)}(X^{(\varepsilon)}(t)) - \varphi^{(n)}(X^{(\varepsilon)}(0)) - \int_0^t (A_s^{(\varepsilon)}\varphi^{(n)})(X^{(\varepsilon)}(s))ds$$

is a martingale for finite n with respect to $\mathcal{F}_t$. But

$$(A_s^{(\varepsilon)}\varphi^{(n)})(x) = \varepsilon^{-1}\int (\varphi^{(n)}(x+\varepsilon^{1/2}z) - \varphi^{(n)}(x))\alpha(\bar{\xi}(s)+\varepsilon^{1/2}x,dz) -$$
$$- \varepsilon^{-1/2}(\varphi^{(n)\prime}(x)\cdot b(\bar{\xi}(s))) \to (A_s^{(\varepsilon)}\varphi)(x)$$

by monotone convergence.

By (3.9), (3.10) and (4.4),

$$|A_s^{(\varepsilon)}\varphi^{(n)}(x)| \le \varepsilon^{-1}\varepsilon^{1/2}\int |z|\alpha(\bar{\xi}(s)+\varepsilon^{1/2}x,dz) + K\cdot\varepsilon^{-1/2} \le K\cdot\varepsilon^{-1/2} .$$

Thus bounded convergence gives

$$\lim_{n\to\infty} G_{t,s}^{(\varepsilon)}A_s^{(\varepsilon)}\varphi^{(n)} = G_{t,s}^{(\varepsilon)}A_s^{(\varepsilon)}\varphi \quad \text{for all} \quad s \ge t$$

and

$$\lim_{n\to\infty}\int_t^\tau G_{t,s}^{(\varepsilon)}A_s^{(\varepsilon)}\varphi^{(n)}ds = \int_t^\tau G_{t,s}^{(\varepsilon)}A_s^{(\varepsilon)}\varphi\, ds .$$

Finally, by monotone convergence $\lim_{n\to\infty} G_{t,\tau}^{(\varepsilon)}\varphi^{(n)} = G_{t,\tau}^{(\varepsilon)}\varphi$ so that

$$G_{t,\tau}^{(\varepsilon)}\varphi - \varphi = \int_t^\tau G_{t,s}^{(\varepsilon)}A_s^{(\varepsilon)}\varphi\, ds .$$

Further,

$$|(G_{t,\tau}^{(\varepsilon)}\varphi)(x)| \le \varphi(x) + K(\tau-t)\varepsilon^{-1/2} \tag{4.5}$$

and

$$E|Z(t)| \le E\varphi(X^{(\varepsilon)}(t)) + E\varphi(X^{(\varepsilon)}(0)) + E\int_0^t |(A_s^{(\varepsilon)}\varphi)(X^{(\varepsilon)}(s))|ds \le$$
$$\le E(\varphi(X^{(\varepsilon)}(t)) - \varphi(X^{(\varepsilon)}(0))) + 2E\varphi(X^{(\varepsilon)}(0)) + t\cdot K\cdot\varepsilon^{-1/2} \le$$
$$\le E\int_0^t |(A_s^{(\varepsilon)}\varphi)(X^{(\varepsilon)}(s))ds + 2K + t\cdot K\cdot\varepsilon^{-1/2} \le$$
$$\le 2\cdot K\cdot t\cdot\varepsilon^{-1/2} + 2K < \infty \quad \text{for all} \quad t < \infty ,$$

so that $Z(t)$ is a martingale

(iii) $\varphi(x) = |x|^2 = \sum_k x_k^2$; $\varphi_k'(x) = 2x_k$.

Choose a sequence $\varphi^{(n)}(x) = |x|^2/(1+|x|^2/n^2) \nearrow \varphi(x)$. $0 \le \varphi^{(n)}(x) \le n^2$.

$$\varphi_k^{(n)\prime}(x) = n^4 \cdot 2x_k/(n^2+|x|^2)^2 \to \varphi_k'(x).$$

$$|\varphi^{(n)\prime}(x)| \le 2|x| \cdot n^4/(n^2+|x|^2)^2 \le 2|x|. \tag{4.6}$$

$$|\varphi^{(n)\prime}(x)| \le n \cdot (2n|x|/(n^2+|x|^2)) \cdot (n^2/(n^2+|x|^2)) \le n,$$

so that $\varphi^{(n)} \in C^1(R^d)$ for all $n < \infty$ and

$$Z^{(n)}(t) = \varphi^{(n)}(X^{(\varepsilon)}(t)) - \varphi^{(n)}(X^{(\varepsilon)}(0)) - \int_0^t (A_s^{(\varepsilon)}\varphi^{(n)})(X^{(\varepsilon)}(s))ds$$

is a martingale with respect to $\mathcal{F}_t$ for all finite n by (i). Further,

$$|\varphi^{(n)}(x+y) - \varphi^{(n)}(x)| =$$

$$= n^2 |(|x+y|^2/(n^2+|x+y|^2)) - (|x|^2/(n^2+|x|^2))| =$$

$$= \big||x+y|^2 - |x|^2\big| \cdot n^4/((n^2+|x+y|^2)(n^2+|x|^2)) \le |y|^2 + 2|x||y|. \tag{4.7}$$

Now $\lim_{n\to\infty} (A_s^{(\varepsilon)}\varphi^{(n)})(x) = (A_s^{(\varepsilon)}\varphi)(x)$ by monotone convergence.
(4.6) and (4.7) in combination with (3.8), (3.9) and (3.10) gives

$$|(A_s^{(\varepsilon)}\varphi^{(n)})(x)| \le \varepsilon^{-1} \cdot \varepsilon^{1/2} \int |z|(2|x| + \varepsilon^{1/2}|z|)\alpha(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) +$$

$$+ \varepsilon^{-1/2}|b(\bar{\xi}(s))| \cdot |\varphi^{(n)\prime}(x)| \le K \cdot |x|\varepsilon^{-1/2} + K + K|x|\,\varepsilon^{-1/2} \le$$

$$\le K(\varepsilon) \cdot (1+|x|^2)^{1/2} = K(\varepsilon)\,\varphi_1(x), \tag{4.8}$$

where by (4.5) $|(G_{t,s}^{(\varepsilon)}\varphi_1)(x)| \le \varphi_1(x) + K\varepsilon^{-1/2}(s-t)$. Then bounded convergence gives

$$\lim_{n\to\infty} G_{t,s}^{(\varepsilon)} A_s^{(\varepsilon)} \varphi^{(n)} = G_{t,s}^{(\varepsilon)} A_s^{(\varepsilon)} \varphi \quad \text{for all} \quad s \ge t$$

and

$$\lim_{n\to\infty} \int_t^\tau G_{t,s}^{(\varepsilon)} A_s^{(\varepsilon)} \varphi^{(n)} ds = \int_t^\tau G_{t,s}^{(\varepsilon)} A_s^{(\varepsilon)} \varphi \, ds.$$

Further, $\lim_{n\to\infty} G_{t,\tau}^{(\varepsilon)} \varphi^{(n)} = G_{t,\tau}^{(\varepsilon)} \varphi$, by monotone convergence, so that

$$G_{t,\tau}^{(\varepsilon)} \varphi - \varphi = \int_t^\tau G_{t,s}^{(\varepsilon)} A_s^{(\varepsilon)} \varphi \, ds.$$

It remains to show that $E|Z(t)| < \infty$, but

$$E(Z(t)| \le E\varphi(X^{(\varepsilon)}(t)) + E\varphi(X^{(\varepsilon)}(0)) +$$

$$+ E\int_0^t |(A_s^{(\varepsilon)}\varphi)(X^{(\varepsilon)}(s))|ds \le E(\varphi(X^{(\varepsilon)}(t)) - \varphi(X^{(\varepsilon)}(0))) +$$

$$+ 2E\varphi(X^{(\varepsilon)}(0)) + E\int_0^t |A_s^{(\varepsilon)}\varphi)(X^{(\varepsilon)}(s))|ds \le$$

$$\le 2\cdot K + 2\cdot K(\varepsilon)\int_0^t E\varphi_1(X^{(\varepsilon)}(s))ds .$$

But by the proof of (ii),

$$E\varphi_1(X^{(\varepsilon)}(t)) \le E|Z_1(t)| + E\varphi_1(X^{(\varepsilon)}(0)) + E\int_0^t |(A_s^{(\varepsilon)}\varphi_1)(X^{(\varepsilon)}(s))|ds \le K(1+t)$$

and thus $E|Z(t)| < \infty$ for all t.

Proof of Lemma 4.4 (ii).

By Lemma 4.5 (iii)

$$Z(t) = \varphi(X^{(\varepsilon)}(t)) - \varphi(X^{(\varepsilon)}(0)) - \int_0^t (A_s^{(\varepsilon)}\varphi)(X^{(\varepsilon)}(s))ds$$

is a martingale for $\varphi(x) = |x|^2$ if $E|X^{(\varepsilon)}(0)|^2 < \infty$. This condition is trivially satisfied when $X^{(\varepsilon)}(0) = x$ so that $Z^*(t) = (Z(t)|X^{(\varepsilon)}(0) = x)$ is a martingale. Thus,

$$0 = E(Z^*(t+h) - Z^*(t)) = E(\varphi(X^{(\varepsilon)}(t+h))|X^{(\varepsilon)}(0) = x) -$$

$$- E(\varphi(X^{(\varepsilon)}(t))|X^{(\varepsilon)}(0) = x) - \int_t^{t+h} E((A_s^{(\varepsilon)}\varphi)(X^{(\varepsilon)}(s))|X^{(\varepsilon)}(0) = x)ds .$$

Here

$$(A_s^{(\varepsilon)}\varphi)(x) = \varepsilon^{-1}\int (|x+\varepsilon^{1/2}z|^2 - |x|^2)\ \alpha(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) -$$

$$- 2\varepsilon^{-1/2}\sum_k x_k b_k(\bar{\xi}(s)) = 2\varepsilon^{-1/2}\sum_k x_k \int z_k \alpha(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) +$$

$$+ \int |z|^2 \alpha(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) - 2\varepsilon^{-1/2}\sum_k x_k b_k(\bar{\xi}(s)) \le$$

$$\le 2\varepsilon^{-1/2}\sum_k x_k(b_k(\bar{\xi}(s) + \varepsilon^{1/2}x) - b_k(\bar{\xi}(s))) + K =$$

$$= 2|x|^2\cdot R(\bar{\xi}(s) + \varepsilon^{1/2}x, \bar{\xi}(s)) + K \le 2\cdot\alpha\varphi(x) + K$$

by the definition of α (4.2).

Let $f(t) = E(\varphi(X^{(\varepsilon)}(t))|X^{(\varepsilon)}(0) = x)$. Then

$$f(t+h) - f(t) \le \int_t^{t+h} (2\alpha f(s) + K)ds = 2\alpha \int_t^{t+h} f(s)ds + K\cdot h .$$

In the same way: $f(t) - f(t-h) \le 2\alpha \int_{t-h}^{t} f(s)ds + K\cdot h$.

By the proof of Lemma 4.5 (iii),

$$f(t) \le K + K(\varepsilon) \int_0^t E(\varphi_1(X^{(\varepsilon)}(s)) \mid X^{(\varepsilon)}(0) = x)ds \le$$

$$\le K + K(\varepsilon)t + K(\varepsilon)t^2 \le K(\varepsilon,T) \quad \text{when} \quad t \le T .$$

By the above $|f(t) - f(s)| \le K|t-s|$ when $s,t \le T$ so that for $t \le T$ $f(t)$ is Lipschitz continuous and $f'(t)$ exists a.e. Further, $f'(t) \le 2\alpha f(t) + K$ as $f(t)$ is continuous for all t. Thus,

$$f(t) \le e^{2\alpha t} f(0) + K(e^{2\alpha t} - 1)/(2\alpha) =$$

$$= e^{2\alpha t} |x|^2 + K(e^{2\alpha t} - 1)/\alpha \le K(t,\alpha)(1 + |x|^2) .$$

As the object is to estimate $D^{(\varepsilon)}_{0,t}(\eta|x) = (D^{(\varepsilon)}_{0,t}\varphi_\eta)(x)$ it would be convenient to use the working formula of Lemma 4.3 on φ_η. This is not possible as $\varphi_\eta \notin C^2(R^d)$. The usual technique in such cases is to smooth φ_η in some way. The most convenient smoothing in this case seems to be Bergströms technique, convolution with a normal distribution with zero mean and "small variance", see [4], [5].

The following lemma is due to Esseen, [8], and Bergström. For the proof, see [4].

Lemma 4.6. Let $\Phi(x)$ be a normal distribution function with density function $(\det C)^{-1/2}\varphi(C^{-1/2}x)$ where $\varphi(x) = (2\pi)^{-d/2} e^{-x^T x/2}$. Let $F(x)$ be a distribution function and $G(x)$ a function which equals $F(x)$ for $x = (+\infty,\dots,+\infty)^T$ and $x = (-\infty,\dots,-\infty)^T$ and is differentiable with $\sup_x |G_i'(x)| \le A\cdot(C_{ii})^{-1/2}$. If $\sup_x |(F(x) - G(x)) * \Phi(x/\delta)| = M(\delta)$, then

$$\sup_x |F(x) - G(x)| \le \max (C_1\cdot M(\delta),\ C_2\cdot\delta\cdot A) .$$

A way of smoothing which is well adapted to Lemma 4.3 is to take

$$\varphi_{h,\eta}(x) = (G_{t,t+h}\varphi_\eta)(x) .$$

Lemma 4.7. If (4.1) holds then, for $h \le 1$,

$$\sup_{\eta} |D^{(\varepsilon)}_{0,t}(\eta|x)| \le K_1 \sup_{\eta} |(D^{(\varepsilon)}_{0,t}\varphi_{h,\eta})(x)| + K_2 \cdot h^{1/2}(1+|x|^2)\cdot h(t),$$

where $h(t) = K(t,\alpha) + K\cdot t^{-1/2}$.

Proof. Use Lemma 4.6 with $\delta = 1$, $C = C(t,t+h)$, $F(\eta) = G^{(\varepsilon)}_{0,t}(\eta|x)$ and $G(\eta) = G_{0,t}(\eta|x)$. Then

$$\varphi_{h,\eta}(x) = (G_{t,t+h}\varphi_\eta)(x) = P(X(t+h) \le \eta | X(t) = x) =$$

$$= P(X(t+h) - \mu(t,t+h)X(t) \le \eta - \mu(t,t+h)x \,|\, X(t) = x) =$$

$$= P(X(t+h) - \mu(t,t+h)X(t) \le \eta - \mu(t,t+h)x) = \Phi(\eta - \mu(t,t+h)x),$$

as it follows from (2.3) that

$$X(t) - \mu(s,t)X(s) = \int_s^t \mu(r,t)\, C^{1/2}(r)\,dW(r),$$

which is independent of $X(s)$ and has mean zero and covariance matrix $C(s,t)$. $0 \le G_i'(\eta) \le (2\pi)^{-1/2}(C_{ii}(0,t))^{-1/2}$ gives

$$A \ge \max_i ((2\pi)^{-1/2}(C_{ii}(t,t+h)/C_{ii}(0,t))^{1/2}).$$

By Lemma 4.1 (v): $C_{ii}(t,t+h) \le K\cdot h$ for all $\underline{a}$ when $h \le 1$ and by Lemma 4.1 (ix): $C_{ii}(0,t) \ge K(1-e^{-2\bar{a}t})/\bar{a} \ge K(1+t^{-1/2})^{-2}$, so that A can be chosen as

$$A = K\cdot h^{1/2}(1+t^{-1/2}).$$

Further

$$(G^{(\varepsilon)}_{0,t}\varphi_{h,\eta})(x) = \int \Phi(\eta - \mu(t,t+h)v)\, G^{(\varepsilon)}_{0,t}(dv|x) =$$

$$= \Phi(\eta) * G^{(\varepsilon)}_{0,t}(\eta|x) + \int (\Phi(\eta - \mu(t,t+h)v) - \Phi(\eta - v))G^{(\varepsilon)}_{0,t}(dv|x)$$

and

$$(G_{0,t}\varphi_{h,\eta})(x) = \Phi(\eta) * G_{0,t}(\eta|x) + \int (\Phi(\eta - \mu(t,t+h)v) - \Phi(\eta - v))G_{0,t}(dv|x),$$

so that

$$(F(\eta) - G(\eta)) * \Phi(\eta) = (D^{(\varepsilon)}_{0,t}\varphi_{h,\eta})(x) -$$

$$- \int (\Phi(\eta - \mu(t,t+h)v) - \Phi(\eta - v))\, D^{(\varepsilon)}_{0,t}(dv|x).$$

Now Lemma 4.6 gives

$$\sup_{\eta} |D^{(\varepsilon)}_{0,t}(\eta|x)| \le C_1 \cdot \sup_{\eta} |(D^{(\varepsilon)}_{0,t}\varphi_{h,\eta})(x)| + C_2 \cdot A +$$

$$+ C_1 \cdot \sup_{\eta} \left| \int (\Phi(\eta - \mu(t,t+h)v) - \Phi(\eta - v)) D^{(\varepsilon)}_{0,t}(dv|x) \right| .$$

Here

$$|\Phi(\eta - \mu(t,t+h)v - \Phi(\eta - v)| \le \sum_k \sup_x |\Phi'_k(x)| |v_k - \mu_k(t,t+h)v| \le$$

$$\le (2\pi)^{-1/2} \sum_k (C_{kk}(t,t+h))^{-1/2} |\mu(t,t+h) - I| \cdot |v| \le$$

$$\le K \cdot |v| \cdot ((1 - e^{-2\bar{a}h}) / \bar{a})^{-1/2} \cdot (1 - e^{-\underline{a}h}) / \underline{a} \le K \cdot |v| \cdot h^{1/2} ,$$

by Lemma 4.1 (iii) and (ix) as $h \le 1$. Further,

$$\int |v| \, (G^{(\varepsilon)}_{0,t}(dv|x) + G_{0,t}(dv|x)) \le$$

$$\le \int (1 + |v|^2)(G^{(\varepsilon)}_{0,t}(dv|x) + G_{0,t}(dv|x)) \le K(t,\alpha)(1 + |x|^2)$$

by Lemma 4.4. Then

$$\sup_{\eta} |D^{(\varepsilon)}_{0,t}(\eta|x)| \le K_1 \sup_{\eta} |(D^{(\varepsilon)}_{0,t} \varphi_{h,\eta})(x)| + K_2 h^{1/2}(1 + |x|^2)h(t) .$$

§5. The rate of convergence in one dimension

Theorem 5.1. Let $d=1$ and suppose

(i) $\alpha^{(\varepsilon)}(\cdot,\cdot) = \alpha(\cdot,\cdot)$ which is independent of ε.

(ii) The assumptions of §3 are valid.

(iii) There exists a constant c_0 and a measure $\bar{\alpha}(\cdot)$ which in the region $|z| > c_0$ dominates all $\alpha(x,\cdot)$ and satisfies $\int_{|z|>c_0} |z|^3 \, \bar{\alpha}(dz) < \infty$.

(iv) $\alpha'(x,z) = \frac{d}{dx}\alpha(x,z)$ exists, is uniformly bounded and satisfies $\sup_x \int |\alpha'(x,dz)| < \infty$ and $\sup_x \int |z|^3 |\alpha'(x,dz)| < \infty$.

(v) $C'(x)$ exists and is bounded.

(vi) $E|X^{(\varepsilon)}(0)|^2 < \infty$ and there exist constants c_1 and K_1 such that

$$P(|X^{(\varepsilon)}(0) - X(0)| > c_1\varepsilon^{1/2}) \le K_1\varepsilon^{1/2} .$$

Then $\sup_x |P(X^{(\varepsilon)}(t) \le x) - P(X(t) \le x)| \le K_\alpha(t)\cdot\varepsilon^{1/2}$ where

$$K_\alpha(t) = \begin{cases} K(1+t^{-1/2}) & \text{if } \alpha < 0 \\ K(t^{3/2}+t^{-1/2}) & \text{if } \alpha = 0 \\ K(e^{ct}+t^{-1/2}) & \text{if } \alpha > 0 . \end{cases}$$

Here K and c are constants which are independent of t and ε.

Remark 5.1. The constant α, which determines the dependence on t of the rate of convergence, was defined in (4.2). It was introduced by Norman in [13]. For $d=1$, $\alpha = \sup_s \sup_x R(x,\bar{\xi}(s))$ where

$$R(x,y) = \begin{cases} (b(x)-b(y))/(x-y) & \text{if } x \ne y \\ b'(y) & \text{if } x = y . \end{cases}$$

As $b(x) = \int z\alpha(x,dz)$, $\alpha < 0$ is the stable case when the drift is always towards $\bar{\xi}$. From the examples of §7 it will be seen that $\alpha = 0$ often indicates the existence of an absorbing point.

Remark 5.2. Note that there are situations in which the restriction to $|z| > c_0$ in condition (iii) is not superfluous. Let for example $\alpha(x,\cdot)$, $0 < x < 1$, be a discrete measure giving mass $1/2$ to $-x/2$ and $(1-x)/2$

respectively. A dominating measure without restrictions would give mass $1/2$ to all points in $(-1/2, 1/2)$, whereas with $c_0 = 1/2$, $\bar{\alpha} \equiv 0$ would do. Obviously, if the measures have uniformly compact support, which is often the case in applications, then condition (iii) is automatically satisfied.

Proof of Theorem 5.1. The first object is to estimate $(D^{(\varepsilon)}_{0,t}\varphi_{h,\eta})(x)$ by means of Lemma 4.3. Using $G_{s,t}G_{t,t+h} = G_{s,t+h}$ gives

$$(D^{(\varepsilon)}_{0,t}\varphi_{h,\eta})(x) = \int_0^t (G_{0,s}(A^{(\varepsilon)}_s - B_s)G_{s,t+h}\varphi_\eta)(x)ds + \\ + \int_0^t (D^{(\varepsilon)}_{0,s}(A^{(\varepsilon)}_s - B_s)G_{s,t+h}\varphi_\eta)(x)ds = I_1(x) + I_2(x) . \tag{5.1}$$

Write

$$\psi(x) = (G_{s,t+h}\varphi_\eta)(x) = \\ = \int_{-\infty}^{\eta} C^{-1/2}(s,t+h)\cdot\varphi(C^{-1/2}(s,t+h)(y - \mu(s,t+h)x))dy , \tag{5.2}$$

where $\varphi(y) = (2\pi)^{-1/2}\cdot e^{-y^2/2}$. Then

$$((A^{(\varepsilon)}_s - B_s)\psi)(x) = \varepsilon^{-1}\int (\psi(x+\varepsilon^{1/2}z) - \psi(x))\,\alpha(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) -$$

$$- \varepsilon^{-1/2}\cdot\psi'(x)\cdot b(\bar{\xi}(s)) - \psi'(x)\cdot B(s)\cdot x - \frac{1}{2}\psi''(x)\cdot C(s) =$$

$$= \varepsilon^{-1}\int (\psi(x+\varepsilon^{1/2}z) - \psi(x) - \varepsilon^{1/2}\cdot z\cdot\psi'(x) - \frac{\varepsilon}{2}z^2\cdot\psi''(x))\alpha(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) +$$

$$+ \varepsilon^{-1/2}\cdot\psi'(x)\cdot(b(\bar{\xi}(s) + \varepsilon^{1/2}x) - b(\bar{\xi}(s)) - \varepsilon^{1/2}\cdot x\cdot B(\bar{\xi}(s))) +$$

$$+ \frac{1}{2}\psi''(x)(C(\bar{\xi}(s) + \varepsilon^{1/2}x) - C(\bar{\xi}(s))) = D_1(x) + D_2(x) + D_3(x) . \tag{5.3}$$

Let $A = C^{-1/2}(s,t+h)\cdot\mu(s,t+h)$ and write $\alpha(\varepsilon^{1/2}x, dz)$ instead of $\alpha(\bar{\xi}(s) + \varepsilon^{1/2}x, dz)$ for convenience. Observe that

$$\sup_x \psi(x) \le 1 , \quad \sup_x |\psi'(x)| \le K\cdot A , \\ \sup_x |\psi''(x)| \le K\cdot A^2 , \quad \sup_x |\psi'''(x)| \le K\cdot A^3 . \tag{5.4}$$

The estimates of I_1 and I_2 of (5.1) are given in the following two lemmas.

Lemma 5.1. Under the conditions of Theorem 5.1, for $h \le 1$,

$$|I_1| \le K\cdot\varepsilon^{1/2}(1+(\varepsilon/h)^{1/2})(K^{3/2}(t,\alpha)+t^{-1/2})(1+x^2). \qquad (5.5)$$

Proof. The estimation is splitted into three parts using (5.3). Let $B = C^{-1/2}(0,s)$ and $x_s = \mu(0,s)\cdot x$.

a) Estimation of $\int_0^t |(G_{0,s}D_1)(x)|ds$.

$$D_1(x) = \varepsilon^{1/2}\int \alpha(\varepsilon^{1/2}x,dz)\, z^3 \int_0^1 (1-t)^2/2 \cdot \psi'''(x+t\,\varepsilon^{1/2}z)dt.$$

By (iii): $\sup_x \int |z|^3\cdot\alpha(x,dz) < \infty$, so that

$$|(G_{0,s}D_1)(x)| = \varepsilon^{1/2}\left|\int dv\, D_1(v)\, B\cdot\varphi(B(v-x_s))\right| \le$$

$$\le \varepsilon^{1/2}\int_0^1 dt(1-t)^2/2 \int dv \int \alpha(\varepsilon^{1/2}v,dz)|z|^3|\psi'''(v+t\,\varepsilon^{1/2}z)|\cdot B\cdot\varphi(B(v-x_s)) \le$$

$$\le K\cdot\varepsilon^{1/2}A^3. \qquad (5.6)$$

$$\text{Let } \chi_N(x) = \begin{cases} 1 & \text{when } |x| \le N \\ (e-e^{|x|-N})/(e-1) & \text{when } N < |x| \le N+1 \\ 0 & \text{when } |x| > N+1. \end{cases} \qquad (5.7)$$

$$\text{Then } |\chi_N'(x)| \le \begin{cases} e/(e-1) & \text{when } N < |x| \le N+1 \\ 0 & \text{otherwise.} \end{cases}$$

By (iii) N can be chosen so that

$$\sup_x \int_{|z|>N} |z|^3\,\alpha(x,dz) \le A^{-2}.$$

Then

$$|(G_{0,s}D_1)(x)| \le$$

$$\le \varepsilon^{1/2}\left|\int dv \int \alpha(\varepsilon^{1/2}v,dz)z^3\,\chi_N(z)\cdot\int_0^1 dt(1-t)^2/2\,\psi'''(v+t\,\varepsilon^{1/2}z)\det B\,\varphi(B(v-x_s))\right| +$$

$$+ K\,\varepsilon^{1/2}A^3\int dv \int_{|z|>N} |z|^3\,\alpha(\varepsilon^{1/2}v,dz)\cdot B\cdot\varphi(B(v-x_s)) \le T_1 + K\cdot\varepsilon^{1/2}\cdot A. \qquad (5.8)$$

For the estimation of T_1 note that if $f(v,z)$, $f_v'(v,z)$, $f_z'(v,z)$ and $f_{vz}''(v,z)$ belong to L^1 then

$$\int dv \int f_v'(v,z)\,\alpha(v,dz) = -\int dv \int f_{vz}''(v,z)\,\alpha(v,z)dz =$$

$$= -\int dz \int dv\, f''_{vz}(v,z)\ \alpha(v,z) = \int dz \int dv\, f'_z(v,z)\ \alpha'(v,z) =$$

$$= \int dv \int dz\, f'_z(v,z)\ \alpha'(v,z) = -\int dv \int f(v,z)\ \alpha'(v,dz)$$

holds as α satisfies (iv). Then

$$\int dv\, B\, \varphi(B(v-x_s)) \int \alpha(\varepsilon^{1/2}v,dz) z^3 \cdot \chi_N(z) \cdot \psi'''(v+t\varepsilon^{1/2}z) =$$

$$= -\varepsilon^{1/2} \int dv B \varphi(B(v-x_s)) \int \alpha'(\varepsilon^{1/2}v,dz) z^3 \cdot \chi_N(z) \psi''(v+t\varepsilon^{1/2}z) -$$

$$- B \int dv\, B\, \varphi'(B(v-x_s)) \int \alpha(\varepsilon^{1/2}v,dz) z^3 \cdot \chi_N(z) \cdot \psi''(v+t\varepsilon^{1/2}z) =$$

$$= -\varepsilon^{1/2} \int dv\, B\, \varphi(B(v-x_s)) \int \alpha'(\varepsilon^{1/2}v,dz) z^3 \cdot \chi_N(z) \cdot \psi''(v+t\varepsilon^{1/2}z) +$$

$$+ \varepsilon^{1/2}\, B \int dv\, B\, \varphi'(B(v-x_s)) \int \alpha'(\varepsilon^{1/2}v,dz) z^3 \cdot \chi_N(z) \cdot \psi'(v+t\varepsilon^{1/2}z) +$$

$$+ B^2 \int dv B \varphi''(B(v-x_s)) \int \alpha(\varepsilon^{1/2}v,dz) z^3 \cdot \chi_N(z) \cdot \psi'(v+t\varepsilon^{1/2}z)$$

holds by the above as $\varphi(B(v-x_s)) \cdot \chi_N(z) \cdot \psi'(v+t\varepsilon^{1/2}z)$ and its derivatives belong to L^1.

For $d=1$ the truncation with $\chi_N(z)$ is actually superfluous as $\psi' = K \cdot \varphi$ so that $\varphi\psi' \in L^1(R^2)$ and $\chi_N(z)$ is not used in the actual estimation of T_1. For $d>1$, however, ψ' does not belong to $L^1(R^d)$ so that $\chi_N(z)$ is needed as an integrating factor.

Now

$$T_1 \le K \cdot \varepsilon \cdot A^2 \int dv B \varphi(B(v-x_s)) \cdot \sup_y \int |z|^3 |\alpha'(y,dz)| +$$

$$+ K \cdot \varepsilon \cdot AB \int dv\, B |\varphi'(B(v-x_s))| \cdot \sup_y \int |z|^3 |\alpha'(y,dz)| +$$

$$+ K \cdot \varepsilon^{1/2}\, AB^2 \int dv\, B |\varphi''(B(v-x_s))| \cdot \sup_y \int |z|^3\, \alpha(y,dz) \le$$

$$\le K \cdot \varepsilon^{1/2} (\varepsilon^{1/2} A^2 + \varepsilon^{1/2}\, AB + AB^2), \qquad (5.9)$$

by assumptions (iii) and (iv). Thus, by (5.8) and (5.9),

$$|(G_{0,s} D_1)(x)| \le K \cdot \varepsilon^{1/2} (A + AB^2 + \varepsilon^{1/2}\, AB + \varepsilon^{1/2} A^2). \qquad (5.10)$$

By Lemma 4.1 (vii):

$$|B| \le K \cdot ((1 - e^{-2\bar{a}s})/\bar{a})^{-1/2} \le K(1 + s^{-1/2}) \quad \text{for all } \bar{a}$$

and by Lemma 4.1 (viii)

$$|A| \le K((e^{2\underline{a}(t+h-s)} - 1)\underline{a})^{-1/2} \le \begin{cases} K(t+h-s)^{-1/2} & \text{when } \underline{a} \ge 0 \\ K\, e^{|\underline{a}|(t+h-s)}(t+h-s)^{-1/2} & \text{when } \underline{a} < 0, \end{cases}$$

so that for $t > 2$, using (5.6) and (5.10),

$$\int_0^t |(G_{0,s}D_1)(x)|ds \le K\cdot\varepsilon^{1/2}\int_0^{t-1} A^3 ds + K\cdot\varepsilon^{1/2}\int_{t-1}^t (A + AB^2 + \varepsilon^{1/2}AB + \varepsilon^{1/2}A^2)ds \le$$

$$\le K\cdot\varepsilon^{1/2}\,(1+(\varepsilon/h)^{1/2})\cdot K^{1/2}(t,\alpha),$$

as

$$\int_0^{t-1} A^3 ds \le \left\{ \begin{matrix} K & \text{when } \underline{a} \ge 0 \\ K\cdot e^{|\underline{a}|t} & \text{when } \underline{a} < 0 \end{matrix} \right\} \le K^{1/2}(t,-\underline{a}) \le K^{1/2}(t,\alpha)$$

and, for $1 \le t-1 \le s \le t$,

$$B \le K, \quad A \le K\cdot h^{-1/2} \quad \text{and} \quad \int_{t-1}^t A\, ds \le K.$$

For $t \le 2$:

$$A \le K(t+h-s)^{-1/2} \le K h^{-1/2}, \quad B \le K s^{-1/2}$$

and

$$\int_0^t |(G_{0,s}D_1)(x)|ds \le K\varepsilon^{1/2}\int_0^{t/2} A^3 ds + K\varepsilon^{1/2}\int_{t/2}^t (A + AB^2 + \varepsilon^{1/2}AB + \varepsilon^{1/2}A^2)ds \le$$

$$\le K\varepsilon^{1/2}(1+(\varepsilon/h)^{1/2})\,(1+t^{-1/2}),$$

as

$$\int_0^{t/2} A^3 ds \le \int_0^{t/2} (t-s)^{-3/2} ds \le K t^{-1/2},$$

$$\int_0^{t/2} A\, ds \le K, \quad \int_0^{t/2} B\, ds \le K$$

and

$$\int_0^{t/2} AB^2 ds \le K t^{-1/2}\int_0^{t/2} (t-s)^{-1/2} s^{-1/2} ds \le K t^{-1/2}.$$

Finally, for all t,

$$\int_0^t |(G_{0,s}D_1)(x)|ds \le K\varepsilon^{1/2}(1+(\varepsilon/h)^{1/2})(K^{1/2}(t,\alpha) + t^{-1/2}). \qquad (5.11)$$

b) Estimation of $\int_0^t |(G_{0,s}D_2)(x)|ds$.

By (3.5), (5.4) and Lemma 4.4 (i):

$$|D_2(x)| \le K\cdot\varepsilon^{1/2}\, A\cdot x^2 \tag{5.12}$$

and

$$|(G_{0,s}D_2)(x)| \le K\cdot\varepsilon^{1/2}\cdot A\cdot\int v^2\, G_{0,s}(dv|x) \le$$
$$\le K\cdot\varepsilon^{1/2}\, A\cdot K(s,\alpha)\,(1+x^2)\,,$$

so that

$$\int_0^t |(G_{0,s}D_2)(x)|ds \le K\cdot\varepsilon^{1/2}\cdot K(t,\alpha)(1+x^2)\cdot\int_0^t A\, ds \le$$
$$\le K\cdot\varepsilon^{1/2}\, K^{3/2}(t,\alpha)\,(1+x^2)\,. \tag{5.13}$$

c) Estimation of $\int_0^t |(G_{0,s}D_3)(x)|ds$.

By (3.6), (5.4) and (v)

$$|(G_{0,s}D_3)(x)| \le \tfrac{1}{2}\left|\int \psi''(v)(C(\bar\xi(s)+\varepsilon^{1/2}v) - C(\bar\xi(s)))\cdot B\cdot\varphi(B(v-x_s))dv\right| \le$$
$$\le \left|\int \psi'(v)(C(\bar\xi(s)+\varepsilon^{1/2}v) - C(\bar\xi(s)))\cdot B^2\cdot\varphi'(B(v-x_s))dv\right| +$$
$$+ \varepsilon^{1/2}\left|\int \psi'(v)\cdot C'(\bar\xi(s)+\varepsilon^{1/2}v)\cdot B\cdot\varphi(B(v-x_s))dv\right| \le$$
$$\le K\cdot\varepsilon^{1/2}\cdot AB\cdot\int |v|\cdot B\cdot|\varphi'(B(v-x_s))|dv + K\cdot\varepsilon^{1/2}\cdot A \le$$
$$\le K\cdot\varepsilon^{1/2}(AB(B^{-1}+|x_s|) + A)\,. \tag{5.14}$$

Now $|x_s| \le |\mu(0,s)|\,|x| \le K\cdot e^{-as}(1+x^2)$ by Lemma 4.1 (ii) and $\int_0^t AB\, ds \le K^{1/2}(t,\alpha)$, so that

$$\int_0^t |(G_{0,s}D_3)(x)|ds \le K\cdot\varepsilon^{1/2}\, K(t,\alpha)\,(1+x^2)\,, \tag{5.15}$$

which, together with (5.11) and (5.13), gives the lemma.

Remark 5.3. Changing to vector notations it is easily seen that the proof of Lemma 5.1 still holds for $d > 1$. The only complication is that $B\cdot B^{-1}$ in (5.14) must be replaced by $|B|\cdot|B^{-1}|$, but by Lemma 4.1 (vi) and (vii): $|B|\cdot|B^{-1}| \le K^{1/2}(s,\alpha)$ so that (5.15) still holds.

Lemma 5.2. Under the conditions of Theorem 5.1, for $h \le 1$,

$$|I_2| \le K\cdot\varepsilon^{1/2}\cdot K^{3/2}(t,\alpha)(1+x^2) + K\cdot\varepsilon^{1/2}(1+(\varepsilon/h)^{1/2})\int_0^t (t+h-s)^{-3/2}\sup_\eta |D^{(\varepsilon)}_{0,s}(\eta|x)|ds \tag{5.16}$$

The following lemma will be used in the proof.

Lemma 5.3. Let $\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$. Then

$$\int \sup_{|m| \le c} |\varphi^{(a)}(x-m)|dx \le K_a \cdot (1+c) .$$

Proof. $\varphi^{(a)}(x) = H_a(x) \cdot \varphi(x)$ where $H_a(x)$ is a polynomial in x of degree a, so that $\varphi^{(a)}(x)$ is monotone in x for $|x| > K_a$. This gives, for $x > K_a + c$, $\sup_{|m| \le c} |\varphi^{(a)}(x-m)| = |\varphi^{(a)}(x-c)|$ and, for $x < -(K_a + c)$, $\sup_{|m| \le c} |\varphi^{(a)}(x-m)| = |\varphi^{(a)}(x+c)|$. Thus

$$\int \sup_{|m| \le c} |\varphi^{(a)}(x-c)|dx \le \int_{|x| \le K_a + c} \sup_x |\varphi^{(a)}(x)|dx +$$

$$+ \int_{K_a+c}^{\infty} |\varphi^{(a)}(x-c)|dx + \int_{-\infty}^{-(K_a+c)} |\varphi^{(a)}(x+c)|dx \le K_a'(K_a+c) + K_a'' \le K_a'''(1+c) .$$

Proof of Lemma 5.2. As in Lemma 5.1 the proof is splitted into three parts. Let $B = C^{-1/2}(s,t+h)$ and $\mu = \mu(s,t+h)$.

a) Estimation of $\int_0^t |(D_{0,s}^{(\varepsilon)} D_1)(x)|ds$.

Let first $t+h-s \le 1$ and observe that by Lemma 4.1 (viii)

$$A \ge K((e^{2\bar{a}(t+h-s)} - 1)/\bar{a})^{-1/2} \ge K((e^{2\bar{a}} - 1)/\bar{a})^{-1/2} \ge K > 0 .$$

With χ defined as in (5.7) write

$$\left| \int D_1(v) D_{0,s}^{(\varepsilon)}(dv|x) \right| \le \left| \int D_1(v)\, \chi_A(v)\, D_{0,s}^{(\varepsilon)}(dv|x) \right| +$$

$$+ \left| \int D_1(v)\, (1 - \chi_A(v))\, D_{0,s}^{(\varepsilon)}(dv|x) \right| = T_1 + T_2 . \qquad (5.17)$$

Observing that $|D_1(v)| \le K \cdot \varepsilon^{1/2} A^3$ gives

$$T_2 \le \int_{|v|>A} |D_1(v)|\,|D_{0,s}^{(\varepsilon)}(dv|x)| \le K \cdot \varepsilon^{1/2} A^3 \int_{|v|>A} (G_{0,s}^{(\varepsilon)}(dv|x) + G_{0,s}(dv|x)) \le$$

$$\le K \cdot \varepsilon^{1/2} \cdot A \int v^2 (G_{0,s}^{(\varepsilon)}(dv|x) + G_{0,s}(dv|x)) \le K \cdot \varepsilon^{1/2} A \cdot K(s,\alpha)(1+x^2) \quad (5.18)$$

by Lemma 4.4.

$$T_1 = \left| \int \frac{d}{dv} (D_1(v)\chi_A(v)) D_{0,s}^{(\varepsilon)}(v|x)dv \right| \le$$

$$\leq \sup_{\eta} |D^{(\varepsilon)}_{0,s}(\eta|x)| \cdot \Big(\int |D_1'(v)| \chi_A(v) dv + \int |D_1(v)| \cdot |\chi_A'(v)| dv \Big) \leq$$

$$\leq \sup_{\eta} |D^{(\varepsilon)}_{0,s}(\eta|x)| \cdot \Big(\int |D_1'(v)| \chi_A(v) dv + K \cdot \varepsilon^{1/2} A^3 \Big) . \qquad (5.19)$$

$$D_1(v) = \varepsilon^{1/2} \int \alpha(\varepsilon^{1/2} v, dz) \cdot z^3 \int_0^1 dt (1-t)^2/2 \cdot \psi'''(v + t\varepsilon^{1/2} z) =$$

$$= -\varepsilon^{1/2} \int_0^1 dt (1-t)^2/2 \int dz \; \alpha(\varepsilon^{1/2} v, z) \cdot \frac{d}{dz} (z^3 \cdot \psi'''(v + t\varepsilon^{1/2} z)) .$$

As $\psi''' = \varphi''$ works as an integrating factor the differentiation $\frac{d}{dv}$ may be performed under the integral sign giving

$$D_1'(v) = -\varepsilon \int_0^1 dt (1-t)^2/2 \int dz \; \alpha'(\varepsilon^{1/2} v, z) \frac{d}{dz} (z^3 \cdot \psi'''(v + t\varepsilon^{1/2} z)) -$$

$$- \varepsilon^{1/2} \int_0^1 dt (1-t)^2/2 \int dz \; \alpha(\varepsilon^{1/2} v, z) \frac{d}{dz} (z^3 \psi''''(v + t\varepsilon^{1/2} z)) =$$

$$= -\varepsilon \int_0^1 dt (1-t)^2/2 \int \alpha'(\varepsilon^{1/2} v, dz) \cdot z^3 \cdot \psi'''(v + t\varepsilon^{1/2} z) -$$

$$- \varepsilon^{1/2} \int_0^1 dt (1-t)^2/2 \int \alpha(\varepsilon^{1/2} v, dz) \cdot z^3 \cdot \psi''''(v + t\varepsilon^{1/2} z) .$$

Now $\psi(x) = \Phi(B\eta - Ax)$ so that

$$\psi'''(x) = -A^3 \varphi''(B\eta - Ax) \quad \text{and} \quad \psi''''(x) = A^4 \cdot \varphi'''(B\eta - Ax) .$$

Then, using conditions (iii) and (iv),

$$\int |D_1'(v)| \chi_A(v) dv \leq$$

$$\leq \varepsilon \int_{|v| < A+1} dv \int_0^1 dt (1-t)^2/2 \int |z|^3 \; |\alpha'(\varepsilon^{1/2} v, dz)| \, |\psi'''(v + t\varepsilon^{1/2} z)| +$$

$$+ \varepsilon^{1/2} \cdot A^3 \int dv \int_0^1 dt (1-t)^2/2 \int_{|z| \leq c_o} \alpha(\varepsilon^{1/2} v, dz) |z|^3 \cdot A \cdot |\varphi'''(B\eta - Av - At\varepsilon^{1/2} z)| +$$

$$+ \varepsilon^{1/2} A^3 \int_0^1 dt (1-t)^2/2 \int_{|z| > c_o} |z|^3 \, \bar{\alpha}(dz) \int dv \cdot A \cdot |\varphi'''(Av - B\eta + At\varepsilon^{1/2} z)| \leq$$

$$\leq K \cdot \varepsilon \cdot A^3 \cdot \int_{|v| < A+1} dv \int |z|^3 \; |\alpha'(\varepsilon^{1/2} v, dz)| +$$

$$+ K \cdot \varepsilon^{1/2} A^3 \int dv \int_{|z| \leq c_o} |z|^3 \; \alpha(\varepsilon^{1/2} A^{-1} v, dz) \cdot \sup_{|m| \leq A \cdot \varepsilon^{1/2} \cdot c_o} |\varphi'''(v - m)| +$$

$$+ K \cdot \varepsilon^{1/2} \cdot A^3 \cdot \int |\varphi'''(v)| dv \leq K \cdot \varepsilon^{1/2} A^3 (1 + (\varepsilon/h)^{1/2}) .$$

using Lemma 5.2 and $K \le A \le K \cdot h^{-1/2}$. Inserting this into (5.19) gives, for $t+h-s \le 1$,

$$\left| \int D_1(v)\, D^{(\varepsilon)}_{0,s}(dv|x) \right| \le K \cdot \varepsilon^{1/2} \cdot A \cdot K(t,\alpha)\ (1+x^2) +$$

$$+ K \cdot \varepsilon^{1/2} (1 + (\varepsilon/h)^{1/2}) (t+h-s)^{-3/2} \cdot \sup_{\eta} \left| D^{(\varepsilon)}_{0,s}(\eta|x) \right| . \qquad (5.20)$$

For $t+h-s > 1$: $A \le K$, which gives $|D_1(x)| \le K \cdot \varepsilon^{1/2} A^3 \le K \cdot \varepsilon^{1/2} A$, so that (5.20) holds for all $s \le t$.

Finally $\int_0^t A\, ds \le K^{1/2}(t,\alpha)$, so that

$$\int_0^t \left| (D^{(\varepsilon)}_{0,s} D_1)(x) \right| ds \le K \cdot \varepsilon^{1/2}\ K^{3/2}(t,\alpha)\ (1+x^2) +$$

$$+ K \cdot \varepsilon^{1/2} (1 + (\varepsilon/h)^{1/2}) \int_0^t (t+h-s)^{-3/2} \cdot \sup_{\eta} \left| D^{(\varepsilon)}_{0,s}(\eta|x) \right| ds . \qquad (5.21)$$

b) Estimation of $\int_0^t \left| (D^{(\varepsilon)}_{0,s} D_2)(x) \right| ds$.

$|D_2(x)| \le K \cdot \varepsilon^{1/2} A \cdot x^2$ by (5.12), so that $\left| (D^{(\varepsilon)}_{0,s} D_2)(x) \right| \le$

$\le K \cdot \varepsilon^{1/2} A \int v^2 (G^{(\varepsilon)}_{0,s}(dv|x) + G_{0,s}(dv|x)) \le K \cdot \varepsilon^{1/2} \cdot A \cdot K(s,\alpha)(1+x^2)$ by Lemma 4.4 and

$$\int_0^t \left| (D^{(\varepsilon)}_{0,s} D_2)(x) \right| ds \le K \cdot \varepsilon^{1/2}\ K^{3/2}(t,\alpha)(1+x^2) . \qquad (5.22)$$

c) Estimation of $\int_0^t \left| (D^{(\varepsilon)}_{0,s} D_3)(x) \right| ds$.

Let first $t+h-s \le 1$ and note that $|D_3(v)| \le K \cdot \varepsilon^{1/2} A^2 |v|$.

$$\left| \int D_3(v)\, D^{(\varepsilon)}_{0,s}(dv|x) \right| \le \left| \int D_3(v)\, \chi_A(v)\, D^{(\varepsilon)}_{0,s}(dv|x) \right| +$$

$$+ \left| \int D_3(v)\, (1-\chi_A(v))\, D^{(\varepsilon)}_{0,s}(dv|x) \right| \le$$

$$\le \sup_{\eta} \left| D^{(\varepsilon)}_{0,s}(\eta|x) \right| \cdot \left\{ \int |D_3'(v)| \cdot \chi_A(v) dv + \int |D_3(v)|\ |\chi_A'(v)| dv \right\} +$$

$$+ K \cdot \varepsilon^{1/2} \cdot A^2 \int_{|v|>A} |v|\ \left| D^{(\varepsilon)}_{0,s}(dv|x) \right| \le$$

$$\le \sup_{\eta} \left| D^{(\varepsilon)}_{0,s}(\eta|x) \right| \left\{ \frac{1}{2} \int_{|v|<A+1} |\psi'''(v)|\ \left| C(\bar{\xi}(s) + \varepsilon^{1/2} v) - C(\bar{\xi}(s)) \right| dv + \right.$$

$$+ \frac{1}{2} \varepsilon^{1/2} \int_{|v|\le A+1} |\psi''(v)|\ \left| C'(\bar{\xi}(s) + \varepsilon^{1/2} v) \right| dv + K \cdot \varepsilon^{1/2} A^3 \Big\} +$$

$$+ K\cdot\varepsilon^{1/2}\cdot A\cdot\int v^2\,|D^{(\varepsilon)}_{0,s}(dv|x)| \le$$

$$\le \sup_\eta |D^{(\varepsilon)}_{0,s}(\eta|x)|\cdot\{K\cdot\varepsilon^{1/2}A^2 \int_{|v|<A+1} A\cdot|\varphi''(Av-B\eta)|\,|v|dv +$$

$$+ K\cdot\varepsilon^{1/2}A^3\} + K\cdot\varepsilon^{1/2}A\,K(s,\alpha)\cdot(1+x^2) \le$$

$$\le K\cdot\varepsilon^{1/2}A\cdot K(t,\alpha)(1+x^2) + K\cdot\varepsilon^{1/2}(t+h-s)^{-3/2}\cdot\sup_\eta |D^{(\varepsilon)}_{0,s}(\eta|x)|. \quad (5.23)$$

For $t+h-s \ge 1$: $|D_3(x)| \le K\cdot\varepsilon^{1/2}A^2|x| \le K\cdot\varepsilon^{1/2}A\,(1+x^2)$ and $|(D^{(\varepsilon)}_{0,s}D_3)(x)| \le K\cdot\varepsilon^{1/2}A\,K(t,\alpha)(1+x^2)$, so that (5.23) holds for all $s \le t$ giving

$$\int_0^t |(D^{(\varepsilon)}_{0,s}D_3)(x)|ds \le K\cdot\varepsilon^{1/2}\,K^{3/2}(t,\alpha)(1+x^2) +$$

$$+ K\cdot\varepsilon^{1/2}\int_0^t (t+h-s)^{-3/2}\sup_\eta |D^{(\varepsilon)}_{0,s}(\eta|x)|ds,$$

which in combination with (5.21) and (5.22) gives the lemma.

Proof of Theorem 5.1 (continued):

By Lemma 4.7:

$$\sup_\eta |D^{(\varepsilon)}_{0,t}(\eta|x)| \le K_1 \sup_\eta |(D^{(\varepsilon)}_{0,t}\varphi_{h,\eta})(x)| + K_2\cdot h^{1/2}(K(t,\alpha)+t^{-1/2})(1+x^2) \quad (5.24)$$

where, by Lemma 5.1 and Lemma 5.2,

$$\sup_\eta |(D^{(\varepsilon)}_{0,t}\varphi_{h,\eta})(x)| \le K\cdot\varepsilon^{1/2}(1+(\varepsilon/h)^{1/2})(K^{3/2}(t,\alpha)+t^{-1/2})(1+x^2) +$$

$$+ K\cdot\varepsilon^{1/2}(1+(\varepsilon/h)^{1/2})\cdot\int_0^t (t+h-s)^{-3/2}\cdot\sup_\eta |D^{(\varepsilon)}_{0,s}(\eta|x)|ds.$$

Inserting this into (5.24) and writing

$$I(t) = \int_0^t (t+h-s)^{-3/2}\sup_\eta |D^{(\varepsilon)}_{0,s}(\eta|x)|ds$$

gives

$$\sup_\eta |D^{(\varepsilon)}_{0,t}(\eta|x)| \le K\cdot\varepsilon^{1/2}(1+(\varepsilon/h)^{1/2})(K^{3/2}(t,\alpha)+t^{-1/2})(1+x^2) +$$

$$+ K\cdot\varepsilon^{1/2}(1+(\varepsilon/h)^{1/2})\cdot I(t) + Kh^{1/2}(K(t,\alpha)+t^{-1/2})(1+x^2). \quad (5.25)$$

Inserted into $I(t)$ this gives

$$I(t) \le C_1 \cdot \varepsilon^{1/2}(1+(\varepsilon/h)^{1/2})(1+x^2)\int_0^t (t+h-s)^{-3/2}(K^{3/2}(s,\alpha)+s^{-1/2})ds +$$

$$+ C_2 \cdot h^{1/2}(1+x^2)\int_0^t (t+h-s)^{-3/2}(K(s,\alpha)+s^{-1/2})ds +$$

$$+ C_3 \cdot \varepsilon^{1/2}(1+(\varepsilon/h)^{1/2})\int_0^t (t+h-s)^{-3/2} I(s)ds \le$$

$$\le C_4 \cdot (1+(\varepsilon/h)^{1/2})^2 (1+x^2)(K^{3/2}(t,\alpha)+t^{-1/2}) +$$

$$+ C_5 \cdot (\varepsilon/h)^{1/2}(1+(\varepsilon/h)^{1/2}) \cdot I(t) ,$$

as

$$\int_0^t (t+h-s)^{-3/2} I(s)ds = \int_0^t (t+h-s)^{-3/2} \int_0^s (s+h-r)^{-3/2} \sup_\eta \left| D^{(\varepsilon)}_{0,r}(\eta|x)\right| drds =$$

$$= \int_0^t \sup_\eta \left| D^{(\varepsilon)}_{0,r}(\eta|x)\right| \int_r^t (t+h-s)^{-3/2}(s+h-r)^{-3/2} dsdr \le$$

$$\le 4 \cdot h^{-1/2} \int_0^t (t+h-r)^{-3/2} \cdot \sup_\eta \left| D^{(\varepsilon)}_{0,r}(\eta|x)\right| dr = 4 \cdot h^{-1/2} I(t)$$

Choose $h = K_0 \cdot \varepsilon$ with K_0 sufficiently large so that $C_6 = C_5 \cdot K_0^{-1/2}(1+K_0^{-1/2}) < 1$. Then

$$I(t) \le (1-C_6)^{-1} \cdot C_4 \cdot (1+K_0^{-1})^2 (1+x^2)(K^{3/2}(t,\alpha)+t^{-1/2}) \le$$

$$\le K(K^{3/2}(t,\alpha)+t^{-1/2})(1+x^2)$$

and, by (5.25),

$$\sup_\eta \left| D^{(\varepsilon)}_{0,t}(\eta|x)\right| \le K \cdot \varepsilon^{1/2}(K^{3/2}(t,\alpha)+t^{-1/2})(1+x^2) \qquad (5.26)$$

if $h = K_0 \cdot \varepsilon \le 1$, but else the theorem is trivially true.

To finish the proof, let $F(x,y)$ be the simultaneous distribution function of $X^{(\varepsilon)}(0)$ and $X(0)$. Then

$$\sup_\eta \left| P(X^{(\varepsilon)}(t) \le \eta) - P(X(t) \le \eta)\right| =$$

$$= \sup_\eta \left| \int (G^{(\varepsilon)}_{0,t}(\eta|x) - G_{0,t}(\eta|y))F(dx,dy)\right| \le$$

$$\le \sup_\eta \left| \int_{|x-y|>c_1\varepsilon^{1/2}} (G^{(\varepsilon)}_{0,t}(\eta|x) - G_{0,t}(\eta|y))F(dx,dy)\right| +$$

$$+ \sup_{\eta} \Big| \int_{|x-y| \le c_1 \varepsilon^{1/2}} (G^{(\varepsilon)}_{0,t}(\eta,x) - G_{0,t}(\eta|y)) F(dx,dy) \Big| \le$$

$$\le K_1 \varepsilon^{1/2} + \int_{|x-y| \le c_1 \varepsilon^{1/2}} \sup_{\eta} \Big| D^{(\varepsilon)}_{0,t}(\eta|x) \Big| \, F(dx,dy) +$$

$$+ \int_{|x-y| \le c_1 \varepsilon^{1/2}} \Big| G_{0,t}(\eta|x) - G_{0,t}(\eta|y) \Big| \, F(dx,dy) \le$$

$$\le K_1 \cdot \varepsilon^{1/2} + K \cdot \varepsilon^{1/2} (K^{3/2}(t,\alpha) + t^{-1/2}) \cdot \int (1 + x^2) F(dx,dy) +$$

$$+ C \cdot \varepsilon^{1/2} \cdot \sup_{x} \Big| \frac{d}{dx} G_{0,t}(\eta|x) \Big| \le K \cdot \varepsilon^{1/2} (K^{3/2}(t,\alpha) + t^{-1/2}) ,$$

as

$$\sup_{x} \Big| \frac{d}{dx} G_{0,t}(\eta|x) \Big| \le C^{-1/2}(0,t) \cdot \mu(0,t) \le K((e^{2\underline{a}t} - 1)/\underline{a})^{-1/2} \le$$

$$\le K(1 + t^{-1/2}) \le K(K^{3/2}(t,\alpha) + t^{-1/2}) ,$$

which proves the theorem.

Remark 5.4. Notice that the continued proof does not use the restriction to $d = 1$ otherwise than in notation and the estimates of Lemma 5.1 and Lemma 5.2. However, as was pointed out in Remark 5.3, the proof of Lemma 5.1 holds for $d > 1$. Thus, all that is needed for the d-dimensional theorem is an analogue to Lemma 5.2.

Remark 5.5. In the case where $\alpha(\xi,\cdot)$ is independent of ξ, that is the jumps are i.i.d. random variables, the rate of convergence is given by the Berry-Esseen theorem as $K \cdot t^{-1/2} \cdot \varepsilon^{1/2}$, which indicates that Theorem 5.1 gives the right order of the remainder term as a function of ε. However, in the i.i.d. case $C = 0$ so that $K_\alpha(t) = K(t^{3/2} + t^{-1/2})$ which is worse than the order given by the Berry-Esseen theorem. As will be seen from the examples in §7 the order $K \cdot t^{-1/2}$ can not hold generally when $\alpha = 0$ and it seems hard to find a non-trivial example where this is the right order.

The next theorem generalizes Theorem 5.1 to the case when the measures $\alpha^{(\varepsilon)}$ are allowed to depend on ε.

Theorem 5.2. Let $d=1$ and suppose

(i) There exist functions $b(x)$ and $C(x)$ such that

$$\sup_x |b^{(\varepsilon)}(x) - b(x)| \le K\cdot\varepsilon \quad \text{and} \quad \sup_x |C^{(\varepsilon)}(x) - C(x)| \le K\cdot\varepsilon .$$

(ii) $B(x) = \frac{d}{dx} b(x)$ exists and the assumption of §3 are valid.

(iii) There exists a constant c_0 and a measure $\bar{\alpha}(\cdot)$ which in the region $|z| > c_0$ dominates all $\alpha^{(\varepsilon)}(x,\cdot)$ and satisfies $\int_{|z|>c_0} |z|^3\, \bar{\alpha}(dz) < \infty$.

(iv) $\frac{d}{dx}\alpha^{(\varepsilon)}(x,z)$ exists, is uniformly bounded and satisfies

$$\sup_{x,\varepsilon} \int |\tfrac{d}{dx}\alpha^{(\varepsilon)}(x,dz)| < \infty \quad \text{and} \quad \sup_{x,\varepsilon} \int |z|^3 |\tfrac{d}{dx}\alpha^{(\varepsilon)}(x,dz)| < \infty .$$

(v) $C'(x)$ exists and is bounded.

(vi) $E(X^{(\varepsilon)}(0))^2 < \infty$ and there exist constants c_1 and K_1 such that

$$P(|X^{(\varepsilon)}(0) - X(0)| > c_1\cdot\varepsilon^{1/2}) \le K_1\cdot\varepsilon^{1/2} .$$

Then $\sup_x |P(X^{(\varepsilon)}(t) \le x) - P(X(t) \le x)| \le K_\beta(t)\cdot\varepsilon^{1/2}$ with $K_\beta(t)$ as in Theorem 5.1 and $\beta = \sup_\varepsilon \sup_s \sup_x R^{(\varepsilon)}(x,\bar{\xi}(s))$ where

$$R^{(\varepsilon)}(x,y) = \begin{cases} (b^{(\varepsilon)}(x) - b^{(\varepsilon)}(y))/(x-y) & \text{when } x \ne y \\ \frac{d}{dy} b^{(\varepsilon)}(y) & \text{when } x = y . \end{cases}$$

Proof. Under the conditions above the lemmas of §4 are still true if α, b and C, when necessary, are replaced by $\alpha^{(\varepsilon)}$, $b^{(\varepsilon)}$ and $C^{(\varepsilon)}$ respectively. Note that $|C^{(\varepsilon)}(x)| \le |C^{(\varepsilon)}(x) - C(x)| + |C(x)| \le K\cdot\varepsilon + K \le K$, so that in the proof of Lemma 4.4 (ii):

$$(A_s^{(\varepsilon)}\varphi)(x) = \varepsilon^{-1}\int ((x+\varepsilon^{1/2}z)^2 - x^2)\alpha^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) -$$

$$- 2\varepsilon^{-1/2}x\cdot b^{(\varepsilon)}(\bar{\xi}(s)) = 2\varepsilon^{-1/2}x\cdot(b^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x) - b^{(\varepsilon)}(\bar{\xi}(s))) +$$

$$+ C^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x) \le 2\cdot\beta\cdot\varphi(x) + K$$

as before. Following the proof of Theorem 5.1, (5.3) is replaced by

$$((A_s^{(\varepsilon)} - B_s)\psi)(x) = \varepsilon^{-1} \int (\psi(x+\varepsilon^{1/2}z) - \psi(x))\alpha^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) -$$

$$- \varepsilon^{-1/2}\psi'(x)b^{(\varepsilon)}(\bar{\xi}(s)) - \psi'(x)\cdot B(s)\cdot x - \frac{1}{2}\psi''(x)C(s) =$$

$$= \varepsilon^{-1} \int (\psi(x+\varepsilon^{1/2}z) - \psi(x) - \varepsilon^{1/2} z\,\psi'(x) - \frac{\varepsilon}{2} z^2\psi''(x))\alpha^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) +$$

$$+ \varepsilon^{-1/2}\cdot\psi'(x)\,(b(\bar{\xi}(s) + \varepsilon^{1/2}x) - b(\bar{\xi}(s)) - \varepsilon^{1/2}\,x\,B(\bar{\xi}(s))) +$$

$$+ \frac{1}{2}\psi''(x)\,(C(\bar{\xi}(s) + \varepsilon^{1/2}x) - C(\bar{\xi}(s))) +$$

$$+ \varepsilon^{-1/2}\,\psi'(x)(b^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x) - b(\bar{\xi}(s) + \varepsilon^{1/2}x) + b^{(\varepsilon)}(\bar{\xi}(s)) - b(\bar{\xi}(s))) +$$

$$+ \frac{1}{2}\psi''(x)\,(C^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x) - C(\bar{\xi}(s) + \varepsilon^{1/2}x)) =$$

$$= D_1^{(\varepsilon)}(x) + D_2(x) + D_3(x) + D_4(x) + D_5(x)\,.$$

Consider the estimations of Lemma 5.1 and Lemma 5.2. $D_1^{(\varepsilon)}$ can be estimated as D_1 under the assumptions of the theorem. The estimates of D_2 and D_3 are the same. $|D_4(x)| \le K\cdot\varepsilon^{1/2}\cdot A$ and $|D_5(x)| \le K\cdot\varepsilon\cdot A^2 = K\cdot\varepsilon^{1/2}A\cdot(\varepsilon/h)^{1/2}\cdot K^{1/2}(t,\alpha)$, so that

$$\int_0^t |(G_{0,s}(D_4 + D_5))(x)|ds \le K\cdot\varepsilon^{1/2}\cdot(1 + (\varepsilon/h)^{1/2})\cdot K(t,\alpha)$$

and

$$\int_0^t |(D_{0,s}^{(\varepsilon)}(D_4 + D_5))(x)|ds \le K\cdot\varepsilon^{1/2}(1 + (\varepsilon/h)^{1/2})\,K(t,\alpha)\,.$$

Thus Lemma 5.1 and Lemma 5.2 remain valid so that the proof is completed word by word as that of Theorem 5.1.

Corollary 5.1. If the processes $\xi^{(\varepsilon)}(t)$ and $\bar{\xi}(t)$ are uniformly bounded then, under the assumptions of Theorem 5.2,

$$\sup_x |P(X^{(\varepsilon)}(t) \le x) - P(X(t) \le x)| \le K_\alpha(t)\cdot\varepsilon^{1/2}$$

where $K_\alpha(t)$ is as in Theorem 5.1.

Remark 5.6. For population processes the condition of the corollary is often satisfied as then $\xi^{(\varepsilon)}(t)$ usually represents the proportion of the population that has a certain property.

Proof. $x = \varepsilon^{-1/2}(\xi - \bar{\xi})$ so that

$$|x| \leq K \cdot \varepsilon^{-1/2} .$$

Then, in the proof of Lemma 4.4 (ii), with $\varphi(x) = x^2$,

$$(A_s^{(\varepsilon)}\varphi)(x) = \varepsilon^{-1} \int ((x+\varepsilon^{1/2}z)^2 - x^2)\, \alpha^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x, dz) -$$

$$- 2\varepsilon^{-1/2} x \cdot b^{(\varepsilon)}(\bar{\xi}(s)) = 2\varepsilon^{-1/2} x (b^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x) - b^{(\varepsilon)}(\bar{\xi}(s))) +$$

$$+ c^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x) \leq 2\varepsilon^{1/2} x (b(\bar{\xi}(s) + \varepsilon^{1/2}x) - b(\bar{\xi}(s))) +$$

$$+2\, \varepsilon^{1/2} x (b^{(\varepsilon)}(\bar{\xi}(s) + \varepsilon^{1/2}x) - b(\bar{\xi}(s) + \varepsilon^{1/2}x) + b^{(\varepsilon)}(\bar{\xi}(s)) - b(\bar{\xi}(s))) + K$$

$$\leq 2 \cdot \varphi(x) \cdot \alpha + K, \text{ when } |x| < K \cdot \varepsilon^{-1/2}$$

and as no other values of x can appear, Lemma 4.4 (ii) is true and the corollary follows.

Corollary 5.2. If in the assumptions of Theorem 5.2 $\underline{c}$ of (3.3) is changed to $\underline{c}(T) = \inf_{t \leq T+1} C(t)$, then the theorem still holds, for $t \leq T$, with $K_\beta(t)$ replaced by $K_\beta(t) \cdot (1 + \underline{c}^{-3}(T))$.

Proof. $\underline{c} > 0$ was explicitly used only in the proof of Lemma 4.1 (vii) - (ix) which now read

(vii) $$|c^{-1/2}(s,t)| \leq K \cdot \underline{c}^{-1/2}(T) \cdot ((1 - e^{-2\bar{a}(t-s)}) / \bar{a})^{-1/2}$$

(viii) $$|c^{-1/2}(s,t) \cdot \mu(s,t)| \leq K \cdot \underline{c}^{-1/2}(T)((e^{2\underline{a}(t-s)} - 1) / \underline{a})^{-1/2}$$

(ix) $$\min_i C_{ii}(s,t) \geq \underline{c}(T) \cdot ((1 - e^{-2\bar{a}(t-s)}) / \bar{a}) .$$

Word by word following the previous proofs and noting the implications of the above changes in Lemma 4.1 gives the corollary.

Remark 5.7. Obviously the corollaries also apply to Theorem 5.1 as Theorem 5.2 reduces to Theorem 5.1 when $\alpha^{(\varepsilon)}$ is independent of ε.

§6. The rate of convergence in d-dimensions (d > 1).

Theorem 6.1. Suppose

(i) $\alpha^{(\varepsilon)}(\cdot,\cdot) = \alpha(\cdot,\cdot)$ which is independent of ε.

(ii) The assumptions of §3 are valid.

(iii) For each integer a, $0 \le a \le d$, and each set $I \subset \{1,\dots,d\}$ with $|I| = a$, $I = \{i_1,\dots,i_a\}$, $\alpha_I^{(a)} = \dfrac{\partial^a}{\partial x_{i_1}\cdots\partial x_{i_a}}\alpha(x,z)$ exists, is uniformly bounded and satisfies $\sup_x \int |\alpha_I^{(a)}(x,dz)| < \infty$ and $\sup_x \int |z|^3 |\alpha_I^{(a)}(x,dz)| <$

(iv) For each integer a, $0 \le a \le d$, and each set $I = \{i_1,\dots,i_a\} \subset \{1,\dots,d\}$ there exist a constant c_0 and a measure $\bar{\alpha}_I^{(a)}(\cdot)$ which in the region $|z| > c_0$ dominates all $|\alpha_I^{(a)}(x,\cdot)|$ and satisfies $\int_{|z|>c_0} |z|^3 \bar{\alpha}_I^{(a)}(dz) < \infty$.

(v) The elements of $C(x)$ are d times differentiable with respect to $x_1,\dots,x_d$ with $\sup_x \left|\dfrac{\partial^a}{\partial x_{i_1}\cdots\partial x_{i_a}} C_{ij}(x)\right| < \infty$ for all i,j,a and I.

(vi) $E|X^{(\varepsilon)}(0)|^2 < \infty$ and their exist constants c_1 and K_1 such that $P(|X^{(\varepsilon)}(0) - X(0)| > c_1\varepsilon^{1/2}) \le K_1 \cdot \varepsilon^{1/2}$. Then

$$\sup_x |P(X^{(\varepsilon)}(t) \le x) - P(X(t) \le x)| \le K_\alpha(t)\cdot\varepsilon^{1/2}$$

with $K_\alpha(t)$ as in Theorem 5.1.

Proof. As was pointed out in Remark 5.4 it remains to show a d-dimensional analogue to Lemma 5.2.

Lemma 6.1. Under the conditions of Theorem 6.1

$$|I_2| \le K\cdot\varepsilon^{1/2}\cdot K^{3/2}(t,\alpha)(1+|x|^2) +$$

$$+ K\cdot\varepsilon^{1/2}(1+(\varepsilon/h)^d)\int_0^t (t+h-s)^{-3/2}\cdot\sup_\eta |D_{0,s}^{(\varepsilon)}(\eta|x)|ds$$

for $h \le \delta_0$ where δ_0 was defined in Lemma 4.1 (iv).

The following analogue to Lemma 5.3 will be used in the proof.

Lemma 6.2. Let $\varphi(x) = (2\pi)^{-d/2} \cdot e^{-|x|^2/2}$ and $\varphi^{(a)}(x) = \dfrac{\partial^a}{\partial x_{i_1} \dots \partial x_{i_a}} \varphi(x)$ for any $i_1, \dots, i_a$. Then

$$\int \sup_{|m| \le c} |\varphi^{(a)}(x - m)| dx \le K_a \cdot (1 + c^d) .$$

Proof. $\varphi(x) = \prod_{i=1}^{d} f(x_i)$ where $f(y) = (2\pi)^{-1/2} e^{-y^2/2}$. Let $\mu(i)$ denote the multiplicity of i in $(i_1, \dots, i_a)$, so that $\sum_{i=1}^{d} \mu(i) = a$ and $\varphi^{(a)}(x) = \prod_{i=1}^{d} f^{(\mu(i))}(x_i)$. Then

$$\int \sup_{|m| \le c} |\varphi^{(a)}(x - m)| dx \le \int \prod_{i=1}^{d} \sup_{|m_i| \le c} |f^{(\mu(i))}(x_i - m_i)| dx =$$

$$= \prod_{i=1}^{d} \int \sup_{|m_i| \le c} |f^{(\mu(i))}(x_i - m_i)| \le \prod_{i=1}^{d} K_{\mu(i)} \cdot (1 + c) \le K_a \cdot (1 + c^d)$$

by Lemma 5.3.

Proof of Lemma 6.1.

$$I_2(x) = \int_0^t (D_{0,s}^{(\varepsilon)} (A_s^{(\varepsilon)} - B_s)\psi)(x) ds ,$$

where

$$\psi(x) = \int_{-\infty}^{\eta} \det C^{-1/2}(s,t+h) \, \varphi(C^{-1/2}(s,t+h)(y - \mu(s,t+h)x)) dy =$$

$$= \int_{\{\mu(s,t+h)u \le \eta\}} \det A \, \varphi(A(x - u)) du = F(\eta - \mu(s,t+h)x) , \qquad (6.1)$$

where $A = C^{-1/2}(s,t+h)$ $\mu(s,t+h)$ and $F(y)$ is a distribution function with density $\det C^{-1/2}(s,t+h) \cdot \varphi(C^{-1/2}(s,t+h)y)$.

Let $B = C^{-1/2}(s,t+h)$, $\mu = \mu(s,t+h)$ and write $\alpha(\varepsilon^{1/2} v, dz)$ instead of $\alpha(\bar{\xi}(s) + \varepsilon^{1/2} v, dz)$.

From (6.1) is seen that

$$\sup_x \psi(x) \le 1 , \quad \max_i \sup_x |\psi_i'(x)| \le K \cdot |A| ,$$

$$\max_{i,j} \sup_x |\psi_{ij}''(x)| \le K|A|^2 , \quad \max_{i,j,k} \sup_x |\psi_{ijk}'''(x)| \le K|A|^3 . \qquad (6.2)$$

The analogue to (5.3) is

$$((A_s^{(\varepsilon)} - B_s)\psi)(x) =$$

$$= \varepsilon^{-1} \int (\psi(x+\varepsilon^{1/2}z) - \psi(x) - \varepsilon^{1/2}(\psi'(x)\cdot z) - \frac{\varepsilon}{2}\cdot z^T\psi''(x)z)\alpha(\varepsilon^{1/2}x,dz) +$$

$$+ \varepsilon^{-1/2}(\psi'(x)\cdot(b(\bar{\xi}(s)+\varepsilon^{1/2}x) - b(\bar{\xi}(s)) - \varepsilon^{1/2}(B(\bar{\xi}(s))\cdot x))) +$$

$$+ \frac{1}{2}\cdot \sum_{k,\ell} \psi''(x)(C_{k\ell}(\bar{\xi}(s)+\varepsilon^{1/2}x) - C_{k\ell}(\bar{\xi}(s))) = D_1(x) + D_2(x) + D_3(x) \qquad (6.3)$$

and the proof is splitted into three parts as before.

Introduce first the notations $\Omega = \{1,\dots,d\}$, Ω_b is a subset of b elements from Ω, $\bar{\Omega}_b = \Omega \setminus \Omega_b$, Ω_{ab} is a subset of a elements from Ω_b and $\bar{\Omega}_{ab} = \Omega_b \setminus \Omega_{ab}$. If $S = (i_1,\dots,i_c)$ then $x_S = (x_{i_1},\dots,x_{i_c})$. Further let $N = |A|$ and introduce $\chi_N(x) = \prod_{i=1}^{d} \chi_N(x_i)$, where $\chi_N(x_i)$ is the truncating function of (5.7).

a) Estimation of $\int_0^t |(D_{0,s}^{(\varepsilon)}D_1)(x)|ds$.

$$D_1(x) = \varepsilon^{1/2} \cdot \sum_{i,j,k} \int \alpha(\varepsilon^{1/2}x,dz) z_i z_j z_k \int_0^1 (1-t)^2/2 \cdot \psi'''_{ijk}(x+t\varepsilon^{1/2}z)dt .$$

Let first $(t+h-s) \le \delta_0$ and write $\left| \int D_1(v) D_{0,s}^{(\varepsilon)}(dv|x) \right| \le T_1 + T_2$ as in (5.17), where $T_2 \le \int_{|v|>N} |D_1(v)| \left| D_{0,s}^{(\varepsilon)}(dv|x) \right| \le K\cdot\varepsilon^{1/2}|A|\, K(s,\alpha)(1+|x|^2)$ as before by Lemma 4.4.

To justify the partial integrations that were used to estimate T_1 it is convenient to introduce the truncation $\chi_M(z)$, where M is chosen so that $\sup_x \int_{|z|>M} |z|^3 \alpha(x,dz) \le |A|^{-2}$, which is possible by (iv). The difference from the case $d=1$ depends on the fact that ψ''' is no longer integrable. Write

$$D_1(v) = D_{11}(v) + D_{12}(v),$$

where $D_{11}(v)$ contains the factor $\chi_M(z)$ and $D_{12}(v)$ contains the factor $(1-\chi_M(z))$. Then

$$T_1 = \left| \int D_1(v)\chi_N(v) D_{0,s}^{(\varepsilon)}(dv|x) \right| \le$$

$$\le \left| \int D_{11}(v)\chi_N(v) D_{0,s}^{(\varepsilon)}(dv|x) \right| + \left| \int D_{12}(v)\chi_N(v) D_{0,s}^{(\varepsilon)}(dv|x) \right| \le$$

$$\le T_{11} + T_{12},$$

where $T_{12} \le K\cdot\varepsilon^{1/2}\cdot|A|$ by the choice of M.

The difficult part is the estimation of T_{11} where the partial integrations used in the proof for $d=1$ give rise to expressions considerably more difficult to estimate for $d > 1$.

$$T_{11} = \left| \int D_{11}(v)\chi_N(v) D^{(\varepsilon)}_{0,s}(dv|x) \right| = \left| \int \frac{\partial^d}{\partial v_\Omega} (D_{11}(v)\chi_N(v)) D^{(\varepsilon)}_{0,s}(v|x)dv \right| \leq$$

$$\leq \sup_\eta \left| D^{(\varepsilon)}_{0,s}(\eta|x) \right| \cdot \int \left| \frac{\partial^d}{\partial v_\Omega} (D_{11}(v)\cdot\chi_N(v)) \right| dv \leq$$

$$\leq \sup_\eta \left| D^{(\varepsilon)}_{0,s}(\eta|x) \right| \cdot \sum_{b,\Omega_b} \int \left| \frac{\partial^b}{\partial v_{\Omega_b}} D_{11}(v) \right| \left| \frac{\partial^{d-b}}{\partial v_{\bar\Omega_b}} \chi_N(v) \right| dv . \qquad (6.4)$$

Here

$$\frac{\partial^b}{\partial v_{\Omega_b}} D_{11}(v) = \varepsilon^{1/2} \sum_{a,\Omega_{ab}} \varepsilon^{a/2} \int_0^1 dt(1-t)^2/2 \; J_{ab}(t,v) ,$$

where (6.5)

$$J_{ab}(t,v) = \int \alpha^{(a)}_{\Omega_{ab}} (\varepsilon^{1/2}v, dz) z_i \cdot z_j \cdot z_k \cdot \frac{\partial^{b-a}}{\partial v_{\bar\Omega_{ab}}} \psi'''_{ijk}(v+t\varepsilon^{1/2}z) .$$

Denote the elements of $\bar\Omega_{ab}$: $i_1, i_2, \ldots, i_{b-a}$ and let $\eta(z) = \eta - t\cdot\varepsilon^{1/2}\mu z$. Then

$$\frac{\partial^{b-a}}{\partial v_{\bar\Omega_{ab}}} \psi(v+t\varepsilon^{1/2}z) = \frac{\partial^{b-a}}{\partial v_{\bar\Omega_{ab}}} F(\eta(z) - \mu v) =$$

$$= \sum_{j_1,\ldots,j_{b-a}} \Big(\prod_{\nu=1}^{b-a} \mu_{j_\nu i_\nu} \Big) \cdot F^{(b-a)}_{j_1\ldots j_{b-a}} (\eta(z) - \mu v) ,$$

where F has density $\det B \cdot \varphi(By)$.

Let $c = c(j_1,\ldots,j_{b-a})$ denote the number of distinct elements among $j_1,\ldots,j_{b-a}$ and denote these elements $k_1,\ldots,k_c$ and the remaining elements $k_{c+1},\ldots,k_{b-a}$.

Let $\Omega_c = \{k_1,\ldots,k_c\}$ and f be the number of times $i_\nu = j_\nu$, $\nu=1\ldots b-$ The only values of j_ν, $\nu = 1 \ldots b-a$, apppearing are $k_1 \ldots k_c$. This implies that $f \leq c$, as $i_k \neq i_\ell$ when $k \neq \ell$, and that

$$| \Omega_b \cap \bar\Omega_c | \leq b - f . \qquad (6.6)$$

But by Lemma 4.1 (iii), as $t+h-s \leq \delta_0$,

$$| \mu(s,t+h) - I | \leq K(1 - e^{-\underline{a}(t+h-s)})/\underline{a} \leq K(t+h-s)$$

so that

$$\mu_{ij} \leq \begin{cases} K(t+h-s) & \text{when } i \neq j \\ K & \text{for all } (i,j). \end{cases}$$

Write $c(x) = x_{\Omega_c}$ and $\bar{c}(x) = x_{\bar{\Omega}_c}$. Define $\tilde{y}$ by $c(\tilde{y}) = c(x)$ and $\bar{c}(\tilde{y}) = \bar{c}(y)$. Then

$$F^{(b-a)}_{j_1 \cdots j_{b-a}}(x) = F^{(b-a)}_{k_1 \cdots k_{b-a}}(x) = \frac{\partial^{b-a-c}}{\partial x_{k_{c+1}} \cdots \partial x_{k_{b-a}}} \frac{\partial^c}{\partial c(x)} F(x) =$$

$$= \frac{\partial^{b-a-c}}{\partial x_{k_{c+1}} \cdots \partial x_{k_{b-a}}} \int_{-\infty}^{\bar{c}(x)} \det B \cdot \varphi(B\tilde{y}) d\bar{c}(y) =$$

$$= \sum_{\ell_{c+1} \cdots \ell_{b-a}} \Big(\prod_{\lambda=c+1}^{b-a} B_{\ell_\lambda k_\lambda} \Big) \int_{-\infty}^{\bar{c}(x)} \det B \cdot \varphi^{(b-a-c)}_{\ell_{c+1} \cdots \ell_{b-a}}(B\tilde{y}) d\bar{c}(y) =$$

$$= \sum_{\ell} B_{\ell} \int_{-\infty}^{\bar{c}(x)} \det B \, \varphi^{(b-a-c)}_{\ell}(B\tilde{y}) d\bar{c}(y).$$

Letting $x = \eta(z) - \mu v$ and $c(\tilde{y}) = c(\eta(z))$ gives

$$\Big| \frac{\partial^{b-a}}{\partial v_{\bar{\Omega}_{ab}}} \psi'''_{ijk}(v + t\varepsilon^{1/2} z) \Big| \leq$$

$$\leq K \cdot |A|^3 \sum_{c=0}^{b-a} |B|^{b-a-c} (t+h-s)^{b-a-f} \sum \int_{R^{d-c}} \det B \, |\varphi^{(3+b-a-c)}(B(\tilde{y} - \mu v))| d\bar{c}(y),$$

omitting the subscripts of φ for convenience. Then

$$|J_{a,b}(t,v)| \leq K \cdot |A|^3 \sum_{c=0}^{b-a} \sum_{f=0}^{c} \sum |B|^{b-a-c} (t+h-s)^{b-a-f} J_{abc}(t,v), \tag{6.7}$$

where

$$J_{abc}(t,v) = \int |\alpha^{(a)}_{\Omega_{ab}}(\varepsilon^{1/2} v, dz)| |z|^3 \int_{R^{d-c}} \det B \varphi^{(3+b-a-c)}(B(\tilde{y} - \mu v)) d\bar{c}(y).$$

Further

$$\Big| \frac{\partial^{d-b}}{\partial v_{\bar{\Omega}_b}} \chi_N(v) \Big| = \chi_N(v_{\Omega_b}) \cdot |\chi'_N(v_{\bar{\Omega}_b})| \leq$$

$$\leq K \cdot \chi_N(v_{\Omega_b} \cap \bar{\Omega}_c) \cdot |\chi'_N(v_{\bar{\Omega}_b} \cap \bar{\Omega}_c)| \leq K \cdot h(\bar{c}(v)),$$

where

$$\int_{R^{d-c}} K \cdot h(\bar{c}(v)) d\bar{c}(v) \leq K \cdot N^{|\Omega_b \cap \bar{\Omega}_c|} \leq K \cdot |A|^{b-f} \quad \text{by (6.6)}.$$

Consider

$$\int J_{abc}(t,v)\left|\frac{\partial^{d-b}}{\partial v_{\bar\Omega_b}}\chi_N(v)\right|dv \le$$

$$\le K\int\{dv\cdot h(\bar c(v))\Big(\int_{|z|\le c_o}+\int_{|z|>c_o}\Big)\left|\alpha^{(a)}_{\Omega_{ab}}(\varepsilon^{1/2}v,dz)\right|\cdot|z|^3\cdot$$

$$\cdot\int_{R^{d-c}}\det B\cdot\left|\varphi^{(3+b-a-c)}(B(\tilde y-\mu v))\right|\,d\bar c(y)\}\le$$

$$\le K\int_{R^{d-c}}d\bar c(v)h(\bar c(v))\cdot$$

$$\cdot\{\int_{R^c}dc(v)\int_{R^{d-c}}d\bar c(y)\cdot\det B\cdot\sup_{|z|\le c_o}\left|\varphi^{(3+b-a-c)}(B(\tilde y-\mu v))\right|+$$

$$+\int_{|z|>c_o}\bar\alpha^{(a)}_{\Omega_{ab}}(dz)|z|^3\int_{R^c}dc(v)\int_{R^{d-c}}d\bar c(y)\det B\left|\varphi^{(3+b-a-c)}(B(\tilde y-\mu v))\right|\}\le \quad (6.8)$$

$$\le K\cdot|A|^{b-f}\,(1+(\varepsilon^{1/2}|B|)^d,$$

using, in the second term, the transformation $x=\tilde y-\mu v$, that is $c(x)=c(\eta)-c(\mu v)-t\varepsilon^{1/2}c(\mu z)$ and $\bar c(x)=\bar c(y)-\bar c(\mu v)$, giving the Jacobian

$$\left|\det\begin{pmatrix}-\mu_{\Omega_c} & 0\\ \tilde\mu & I\end{pmatrix}\right|^{-1}=\left|\det\mu_{\Omega_c}\right|^{-1}\le 2$$

by Lemma 4.1 (iv), and the transformation

$c(x)=c(\eta)-c(\mu v)$, $\bar c(x)=\bar c(y)-\bar c(\mu v)$, with the same Jacobian, plus Lemma 6.2 in the first.

Returning via (6.7) and (6.5) to (6.4):

$$T_{11}\le K\cdot\varepsilon^{1/2}(1+(\varepsilon^{1/2}|B|)^d)\cdot|A|^3\cdot\sup_\eta\left|D^{(\varepsilon)}_{0,s}(\eta|x)\right|\cdot(1+(\varepsilon/h)^{d/2})\le$$

$$\le K\,\varepsilon^{1/2}(1+(\varepsilon/h)^d)(t+h-s)^{-3/2}\cdot\sup_\eta\left|D^{(\varepsilon)}_{0,s}(\eta|x)\right|$$

as

$$\sum_{b=0}^{d}\sum_{a=0}^{b}\varepsilon^{a/2}\sum_{c=1}^{b-a}\sum_{f=0}^{c}|B|^{b-a-c}(t+h-s)^{b-a-f}|A|^{b-f}\le$$

$$\le K\cdot\sum_{b=0}^{d}\sum_{a=0}^{b}(\varepsilon/h)^{a/2}\sum_{c=1}^{b-a}(|B||A|(t+h-s))^{b-a-c}\sum_{f=0}^{c}(|A|(t+h-s))^{c-f}\le$$

$$\le K\cdot\sum_{b=0}^{d}\sum_{a=0}^{b}(\varepsilon/h)^{a/2}\le K\cdot(1+(\varepsilon/h)^{d/2}),$$

using $t+h-s \le \delta_0$, $|A| \le K(t+h-s)^{-1/2} \le K\cdot h^{-1/2}$ and $|B| \le K(t+h-s)^{-1/2}$.

Collecting the estimates of T_2, T_{11} and T_{12} gives

$$\left| \int D_1(v)\, D_{0,s}^{(\varepsilon)}(dv|x) \right| \le K\cdot\varepsilon^{1/2}|A|\, K(t,\alpha)\, (1+|x|^2) +$$

$$+ K\cdot\varepsilon^{1/2}(1+(\varepsilon/h)^d)\, (t+h-s)^{-3/2} \cdot \sup_\eta \left| D_{0,s}^{(\varepsilon)}(\eta|x) \right|,$$

for $t+h-s \le \delta_0$, but it is proved exactly as in the case $d=1$ that the inequality holds for all $s \le t$ giving

$$\int_0^t |(D_{0,s}^{(\varepsilon)}D_1)(x)|ds \le K\cdot\varepsilon^{1/2}(1+|x|^2)\cdot K^{3/2}(t,\alpha) +$$

$$+ K\cdot\varepsilon^{1/2}(1+(\varepsilon/h)^d) \int_0^t (t+h-s)^{-3/2} \sup_\eta \left| D_{0,s}^{(\varepsilon)}(\eta|x) \right| ds .$$

b) $$\int_0^t |(D_{0,s}^{(\varepsilon)}D_2)(x)|ds \le K\cdot\varepsilon^{1/2}(1+|x|^2)\cdot K^{3/2}(t,\alpha)$$

is proved exactly as for $d=1$.

c) <u>Estimation of</u> $\int_0^t |(D_{0,s}^{(\varepsilon)}D_3)(x)|ds$.

$$D_3(x) = \frac{1}{2}\cdot\Sigma\, \psi_{ij}''(x)(C_{ij}(\bar{\xi}(s)+\varepsilon^{1/2}x) - C_{ij}(\bar{\xi}(s))) .$$

Let first $t+h-s \le \delta_0$ and write $\left| \int D_3(v) D_{0,s}^{(\varepsilon)}(dv|x) \right| \le T_1 + T_2$, where

$$T_2 = \left| \int D_3(v)(1-\chi_N(v)) D_{0,s}^{(\varepsilon)}(dv|x) \right| \le$$

$$\le K\cdot\varepsilon^{1/2}|A|^2 \int_{|v|>N} |v|\, (G_{0,s}^{(\varepsilon)}(dv|x) + G_{0,s}(dv|x)) \le$$

$$\le K\cdot\varepsilon^{1/2}|A|\, (1+|x|^2)\, K(s,\alpha) .$$

$$T_1 = \left| \int \frac{\partial^d}{\partial v_\Omega} (D_3(v)\chi_N(v))\, D_{0,s}^{(\varepsilon)}(v|x) \right| \le$$

$$\le \sup_\eta \left| D_{0,s}^{(\varepsilon)}(\eta|x) \right| \cdot \sum_{b,\Omega_b} \int \left| \frac{\partial^b}{\partial v_{\Omega_b}} D_3(v) \right| \left| \frac{\partial^{d-b}}{\partial v_{\bar{\Omega}_b}} \chi_N(v) \right| dv \le$$

$$\le \sup_\eta \left| D_{0,s}^{(\varepsilon)}(\eta|x) \right| \cdot \sum_{b,\Omega_b} \sum_{a,\Omega_{ab}} \sum_{i,j} I_{ab} ,$$

where

$$I_{0b} = \int \left|\frac{\partial^b}{\partial v_{\bar\Omega_b}} \psi''_{ij}(v)\right| \left|C_{ij}(\bar\xi(s) + \varepsilon^{1/2}v) - C_{ij}(\bar\xi(s))\right| \left|\frac{\partial^{d-b}}{\partial v_{\bar{\bar\Omega}_b}} \chi_N(v)\right| dv \le$$

$$\le K\cdot\varepsilon^{1/2}(N+1) \int \left|\frac{\partial^b}{\partial v_{\bar\Omega_b}} \psi''_{ij}(v)\right| \left|\frac{\partial^{d-b}}{\partial v_{\bar{\bar\Omega}_b}} \chi_N(v)\right| dv \le K\cdot\varepsilon^{1/2}|A|\, J_{0b}$$

and, for $a > 0$,

$$I_{ab} = \varepsilon^{a/2} \int \left|C^{(a)}_{ij}(\bar\xi(s) + \varepsilon^{1/2}v)\right| \left|\frac{\partial^{b-a}}{\partial v_{\bar{\bar\Omega}_{ab}}} \psi''_{ij}(v)\right| \left|\frac{\partial^{d-b}}{\partial v_{\bar{\bar\Omega}_b}} \chi_N(v)\right| dv \le$$

$$\le K\cdot\varepsilon^{1/2}\cdot\varepsilon^{(a-1)/2}\, J_{ab},$$

where

$$J_{ab} = \int \left|\frac{\partial^{b-a}}{\partial v_{\bar{\bar\Omega}_{ab}}} \psi''_{ij}(v)\right| \left|\frac{\partial^{d-b}}{\partial v_{\bar{\bar\Omega}_b}} \chi_N(v)\right| dv .$$

Here

$$\left|\frac{\partial^{b-a}}{\partial v_{\bar{\bar\Omega}_{ab}}} \psi''_{ij}(v)\right| \le$$

$$\le K\cdot|A|^2\cdot\Sigma\ |B|^{b-a-c}(t+h-s)^{b-a-f} \int_{R^{d-c}} \det B\, \left|\varphi^{(2+b-a-c)}(B(\tilde y - \mu v))\right| d\bar c(y),$$

where $c(\tilde y) = c(\eta)$ and $\bar c(\tilde y) = \bar c(y)$. Thus

$$J_{ab} \le K\cdot|A|^2\cdot\Sigma\ |B|^{b-a-c}(t+h-s)^{b-a-f} \cdot \int_{R^{d-c}} d\bar c(v)\ h(\bar c(v))\cdot I,$$

where

$$I = \int_{R^c} dc(v) \int_{R^{d-c}} d\bar c(y)\ \det B\, \left|\varphi^{(2+b-a-c)}(B(\tilde y - \mu v))\right| \le K$$

as in a), so that

$$J_{ab} \le K|A|^2\,\Sigma\ |B|^{b-a-c}(t+h-s)^{b-a-f}\cdot|A|^{b-f} \le K\cdot|A|^2\cdot|A|^a .$$

Then $I_{0b} \le K\cdot\varepsilon^{1/2}|A|^3$ and $I_{ab} \le K\cdot\varepsilon^{1/2}|A|^3(\varepsilon/h)^{(a-1)/2}$ for $a > 0$, so that

$$T_1 \le K\cdot\sup_\eta \left|D^{(\varepsilon)}_{0,s}(\eta|x)\right|\cdot\varepsilon^{1/2}|A|^3\Big(1 + \sum_{b=1}^{d}\sum_{a=1}^{b} (\varepsilon/h)^{(a-1)/2}\Big) \le$$

$$\le K\cdot\varepsilon^{1/2}(1 + (\varepsilon/h)^{d/2})\cdot\sup_\eta \left|D^{(\varepsilon)}_{0,s}(\eta|x)\right|\cdot(t+h-s)^{-3/2}$$

and

$$\left|(D^{(\varepsilon)}_{0,s}D_3)(x)\right| \le K\cdot\varepsilon^{1/2}\, K(t,\alpha)\,|A|(1 + |x|^2) +$$

$$+ K\cdot\varepsilon^{1/2}\,(1+(\varepsilon/h)^d)\cdot(t+h-s)^{-3/2}\cdot\sup_\eta\left|D^{(\varepsilon)}_{0,s}(\eta|x)\right|$$

for $t+h-s\le\delta_0$, but it is proved exactly as in the one-dimensional case that the inequality holds for all $s\le t$. This gives

$$\int_0^t |(D^{(\varepsilon)}_{0,s}D_3)(x)|ds \le K\cdot\varepsilon^{1/2}\cdot K^{3/2}(t,\alpha)\cdot(1+|x|^2) +$$

$$+ K\cdot\varepsilon^{1/2}\,(1+(\varepsilon/h)^d)\int_0^t (t+h-s)^{-3/2}\sup_\eta\left|D^{(\varepsilon)}_{0,s}(\eta|x)\right|,$$

which proves the lemma and thereby also Theorem 6.1.

Remark 6.1. Theorem 6.1 with $d=1$ is not equivalent to Theorem 5.1. To be so Condition (iv) would have to be changed to hold only for $0\le a\le d-1$. But the condition, for $a=d$, was used only to estimate

$$\int dv \int_{|z|>c_0} |\alpha^{(d)}_\Omega(\varepsilon^{1/2}v,dz)|\,|z|^3\,h(v)\int_{R^d}\det B\,|\varphi^{(3)}(B(y-\mu v))|dy \quad \text{in (6.8)}$$

as $b=d$, $c=0$ and $f=0$ when $a=d$. The expression above is less than $K\cdot|A|^d$ by simply using $\int h(v)dv\le|A|^d$, which proves the following corollary.

Corollary 6.1. Theorem 6.1 still holds if Condition (iv) is changed to hold only for $0\le a\le d-1$.

When the measures $\alpha^{(\varepsilon)}$ are allowed to depend on ε the following analogue to Theorem 5.2 holds.

Theorem 6.2. Suppose

(i) There exist a vector $b(x)$ and a matrix $C(x)$ such that

$$\sup_x|b^{(\varepsilon)}(x)-b(x)|\le K\cdot\varepsilon \quad\text{and}\quad \sup_x|C^{(\varepsilon)}(x)-C(x)|\le K\cdot\varepsilon.$$

(ii) $B(x)=\frac{d}{dx}b(x)$ exists and the assumptions of §3 are valid.

(iii) For each integer a, $0\le a\le d$, and each set $I\subset\{1,\dots,d\}$ with $|I|=a$, $I=\{i_1,\dots,i_a\}$, $\frac{\partial^a}{\partial x_{i_1}\dots\partial x_{i_a}}\alpha^{(\varepsilon)}(x,z)$ exists for all ε, is

$$\sup_{\varepsilon}\sup_{x}\int \left|\frac{\partial^a}{\partial x_{i_1}\cdots\partial x_{i_a}}\,\alpha^{(\varepsilon)}(x,dz)\right| < \infty$$

and

$$\sup_{\varepsilon}\sup_{x}\int |z|^3 \left|\frac{\partial^a}{\partial x_{i_1}\cdots\partial x_{i_a}}\,\alpha^{(\varepsilon)}(x,dz)\right| < \infty .$$

(iv) For each integer a, $0 \le a \le d-1$ and each set $I = \{i_1,\dots,i_a\} \subset \{1,\dots,d\}$ there exist a constant c_0 and a measure $\bar{\alpha}_I^{(a)}$ which in the region $|z| > c_0$ dominates all $\left|\frac{\partial^a}{\partial x_{i_1}\cdots\partial x_{i_a}}\,\alpha^{(\varepsilon)}(x,\cdot)\right|$ and satisfies $\int_{|z|>c_0} |z|^3\,\bar{\alpha}_I^{(a)}(dz) < \infty$.

(v) The elements of $C(x)$ are d times differentiable with respect to $x_1 \dots x_d$ with

$$\sup_{x}\left|\frac{\partial^a}{\partial x_{i_1}\cdots\partial x_{i_a}}\,C_{ij}(x)\right| < \infty \quad \text{for all } i,j,a \text{ and } I .$$

(vi) $E|X^{(\varepsilon)}(0)|^2 < \infty$ and there exist constants c_1 and K_1 such that

$$P(|X^{(\varepsilon)}(0) - X(0)| > c_1\cdot\varepsilon^{1/2}) \le K_1\cdot\varepsilon^{1/2} .$$

Then

$$\sup_{x}|P(X^{(\varepsilon)}(t) \le x) - P(X(t) \le x)| \le K_\beta(t)\,\varepsilon^{1/2},$$

where $\beta = \sup_{\varepsilon}\sup_{s}\sup_{x} R^{(\varepsilon)}(x,\bar{\xi}(s))$ with

$$R^{(\varepsilon)}(x,y) = \begin{cases} \sum_k (b_k^{(\varepsilon)}(x) - b_k^{(\varepsilon)}(y))(x_k - y_k)/|x-y|^2 & \text{if } x \ne y \\ \lambda_d((B^{(\varepsilon)}(y) + (B^{(\varepsilon)}(y))^T)/2) & \text{if } x = y . \end{cases}$$

Proof. Follows immediately by introducing vector notation in the proof of Theorem 5.2.

Corollary 6.2. If the processes $\xi^{(\varepsilon)}(t)$ and $\bar{\xi}(t)$ are uniformly bounded then, under the assumptions of Theorem 6.2,

$$\sup_{x}|P(X^{(\varepsilon)}(t) \le x) - P(X(t) \le x)| \le K_\alpha(t)\cdot\varepsilon^{1/2}$$

with $K_\alpha(t)$ as in Theorem 6.1.

Proof. As that of Corollary 5.1.

Corollary 6.3. If in the assumptions of Theorem 6.2, $\underline{c}$, of (3.3), is changed to $\underline{c}(T) = \inf_{t \leq T+1} \lambda_1(C(t))$ then the theorem holds for $t \leq T$ with $K_\beta(t)$ replaced by $K_\beta(t) \cdot (1 + \underline{c}^{-3}(T))$.

Proof. As that of Corollary 5.2.

§7. Examples

In this chapter two examples from epidemiology and two examples from chemistry will be given. In all these the measures $\alpha^{(\varepsilon)}(x,\cdot)$ are discrete and to avoid repeated verification of the assumptions of the previous theorems two new theorems on discrete measures will be given.

Theorem 7.1. Suppose that the support of the measures $\alpha^{(\varepsilon)}(x,\cdot) \equiv \alpha(x,\cdot)$ is $\{a_1,\ldots,a_k\}$ and that $\alpha(\cdot,\{a_i\}) \in C^d(R^d)$ for $i=1,\ldots,k$ and satisfies a second order Lipschitz condition.

If the initial conditions (vi) of Theorem 6.1 are satisfied, then

$$\sup_x |P(X^{(\varepsilon)}(t) \le x) - P(X(t) \le x)| \le \begin{cases} K_\alpha(t)\varepsilon^{1/2} & \text{if } \underline{c} > 0 \\ K_\alpha(t)(1+\underline{c}^{-3}(T))\varepsilon^{1/2}, & \text{for } t \le T, \text{ if } \underline{c}(T) > 0 \end{cases}$$

with $K_\alpha(t)$ as in Theorem 6.1.

Proof. Check the conditions of Theorem 6.1 and note that Corollary 6.2 applies also to Theorem 6.1. (i) and (vi) are assumed in the theorem.

$\sup_\xi \sum_{i=1}^k \alpha(\xi,\{a_i\}) < \infty$ implies (3.1).

As $b(\xi) = \sum_{i=1}^k a_i \cdot \alpha(\xi,\{a_i\})$, $C(\xi) = \sum_{i=1}^k a_i (a_i)^T \cdot \alpha(\xi,\{a_i\})$ and $B(\xi) = \sum_{i=1}^k a_i \cdot \frac{d}{d\xi} \alpha(\xi,\{a_i\})$ it follows that $\sup_\xi |B(\xi)| < \infty$, proving (3.2) and (3.4), and that $\sup_\xi |C(\xi)| < \infty$, proving (3.7) and the right hand side of (3.3) and as $\underline{c} > 0$ or $\underline{c}(T) > 0$ in the theorem (3.3) holds. The Lipschitz assumption gives (3.5) and (3.6) so that Condition (ii) is satisfied.

In (iii), the existence of $\alpha_I^{(a)}$ is assumed and, as $\alpha(\cdot,\{a_i\}) \in C^d(R^d)$ $\alpha_I^{(a)}$ is uniformly bounded, $\sup_\xi \{|\alpha_I^{(a)}(\xi,dz)|\} = \sup_\xi \sum_{i=1}^k |\alpha_I^{(a)}(\xi,\{a_i\})| < \infty$ and $\sup_\xi \{|z|^3 |\alpha_I^{(a)}(\xi,dz)|\} = \sup_\xi \sum_{i=1}^k |a_i|^3 |\alpha_I^{(a)}(\xi,\{a_i\})| < \infty$ so that (iii) holds.

(iv) is trivially satisfied with $c_0 = \max_{1 \le i \le k} |a_i| + 1$ and $\bar{\alpha}_I^{(a)} \equiv 0$.

Finally, (v) follows as

$$\frac{\partial^a}{\partial\xi_1 \cdots \partial\xi_d} C(\xi) = \sum_{i=1}^k a_i (a_i)^T \cdot \alpha_I^{(a)}(\xi,\{a_i\}) .$$

Theorem 7.2. Suppose that the measures $\alpha^{(\varepsilon)}(x,\cdot)$ have support $\{a_1,\dots,a_k\}$, that $\alpha^{(\varepsilon)}(\cdot,\{a_i\}) \in C^d(R^d)$ uniformly in ε for $i=1,\dots,k$ and that there exist measures $\alpha(x,\cdot)$ with the same support, where $\alpha(\cdot,\{a_i\}) \in C^d(R^d)$ and satisfies a second order Lipschitz condition for $i=1,\dots,k$.

Suppose further that

$$\sup_{\xi} |\alpha^{(\varepsilon)}(\xi,\{a_i\}) - \alpha(\xi,\{a_i\})| \le K\cdot\varepsilon \tag{7.1}$$

and that the processes $\xi^{(\varepsilon)}(t)$ and $\bar{\xi}(t)$ are uniformly bounded.

If the initial conditions (vi) of Theorem 6.1 are satisfied, then

$$\sup_x |P(X^{(\varepsilon)}(t)\le x) - P(X(t)\le x)| \le \begin{cases} K_\alpha(t)\cdot\varepsilon^{1/2} & \text{if } \underline{c}>0 \\ K_\alpha(t)(1+\underline{c}^{-3}(T))\cdot\varepsilon^{1/2}, \text{ for } t\le T, & \text{if } \underline{c}(T)>0 \end{cases}$$

with $K_\alpha(t)$ as in Theorem 6.1.

Proof. Noting that, by (7.1),

$$\sup_{\xi} |b^{(\varepsilon)}(\xi) - b(\xi)| = \sup_{\xi} |\sum_i a_i(\alpha^{(\varepsilon)}(\xi,\{a_i\}) - \alpha(\xi,\{a_i\}))| \le$$

$$\le \sum_i |a_i|\cdot\sup_{\xi} |\alpha^{(\varepsilon)}(\xi,\{a_i\}) - \alpha(\xi,\{a_i\})| \le K\cdot\varepsilon$$

and that, in the same way, $\sup_{\xi} |C^{(\varepsilon)}(\xi) - C(\xi)| \le K\cdot\varepsilon$, the theorem follows from Corollary 6.1 with arguments similar to those in the preceding proof.

Example 7.1. Consider the following one-dimensional epidemic model for diseases without immunity.

In a population of N individuals let $Y^{(N)}(t)$ denote the number of infected individuals at time t, $Y^{(N)}(0) > 0$. With $\varepsilon = N^{-1}$ and $\xi^{(\varepsilon)}(t) = \varepsilon Y^{(N)}(t)$ the infinitesimal transition probabilities are

$$P(Y^{(N)}(t+\Delta t) - Y^{(N)}(t) = 1 \mid Y^{(N)}(t) = y) = k_1\cdot y\cdot\frac{N-y}{N}\cdot\Delta t + O(\Delta t) =$$

$$= \varepsilon^{-1}k_1\cdot\xi\cdot(1-\xi)\cdot\Delta t + O(\Delta t)$$

and

$$P(Y^{(N)}(t+\Delta t) - Y^{(N)}(t) = -1 \mid Y^{(N)}(t) = y) = k_2\cdot y\cdot\Delta t + O(\Delta t) =$$

$$= \varepsilon^{-1}\, k_2\,\xi\cdot\Delta t + O(\Delta t)$$

As $0 \leq \xi \leq 1$ it suffices that $\alpha(\cdot,\{i\}) \in C^1[0,1]$, $i = -1,1$, and satisfies the Lipschitz condition which is easily verified.

$$b(\xi) = k_1 \xi(1 - \xi) - k_2 \xi = (k_1 - k_2)\xi - k_1 \xi^2 .$$

$$C(\xi) = (k_1 + k_2)\xi - k_1 \xi^2 .$$

$$B(\xi) = k_1 - k_2 - 2k_1 \xi .$$

$\bar{\xi}(t)$ is the solution to

$$\begin{cases} \xi'(t) = b(\xi(t)) \\ \xi(0) = \xi_0 . \end{cases}$$

The solutions to $b(\xi) = 0$ are $\xi = 0$ and $\xi = 1 - k_2/k_1$ so that for $k_1 \leq k_2$:

$$\bar{\xi}(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{for all} \quad \xi_0 .$$

In this case $\underline{c} = 0$, $\alpha > 0$ and Theorem 7.1 gives the rate of convergence as

$$K \cdot (e^{ct} + t^{-1/2})(1 + \underline{c}^{-3}(T)) \cdot \varepsilon^{1/2}, \quad \text{for} \quad t \leq T,$$

where $\underline{c}(T) > 0$ if $\xi_0 > 0$.

Let $k_1 > k_2$. Then $\tilde{\xi} = 1 - k_2/k_1$ is an equilibrium point and $\bar{\xi}(t) \to \tilde{\xi}$ as $t \to \infty$ if $\xi_0 > 0$.

$(b(\xi) - b(\tilde{\xi}))/(\xi - \tilde{\xi}) = -k_1 \xi$ so that

$$\bar{\xi}(t) \searrow \tilde{\xi} \quad \text{when} \quad \xi_0 \geq \tilde{\xi}, \quad \text{and then} \quad \alpha = 0,$$

and

$$\bar{\xi}(t) \nearrow \tilde{\xi} \quad \text{when} \quad \xi_0 < \tilde{\xi}, \quad \text{and then} \quad \alpha > 0 .$$

$\underline{c} = \inf_t C(\bar{\xi}(t)) > 0$ in both cases so that Theorem 7.1 is applicable, under suitable initial conditions, giving the rate of convergence as $K(t^{3/2} + t^{-1/2}) \cdot \varepsilon^{1/2}$ for $\xi_0 \geq \tilde{\xi}$ and $K(e^{ct} + t^{-1/2}) \cdot \varepsilon^{1/2}$ for $\xi_0 < \tilde{\xi}$. With $\xi_0 = Y(0)/N$ the initial conditions are trivially satisfied.

If, for different N, we want to study the behaviour for a fixed infected initial proportion ξ_0 then, with $Y^{(N)}(0) = [N \cdot \xi_0]$, $X^{(\varepsilon)}(0) = \varepsilon^{1/2}(Y^{(N)}(0) - N\xi_0)$ whereas $X(0) = 0$ so that $|X^{(\varepsilon)}(0) - X(0)| \leq \varepsilon^{1/2}$

and the initial conditions are satisfied.

Example 7.2. The following two-dimensional epidemic model allows immunity. It is usually referred to as the simple stochastic epidemic.

Consider a population of N individuals where, at time t, $Y_1(t)$ = the number of susceptible and $Y_2(t)$ = the number of infected individuals. $N - Y_1 - Y_2$ is the number of individuals that are immune to the disease.

With $\varepsilon = N^{-1}$ and $\xi^{(\varepsilon)}(t) = \varepsilon Y^{(N)}(t)$, $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$,

$$P(Y^{(N)}(t+\Delta t) - Y^{(N)}(t) = \begin{pmatrix} -1 \\ +1 \end{pmatrix} \mid Y^{(N)}(t) = y) = k_1 \cdot y_1 \cdot \frac{y_2}{N} \cdot \Delta t + O(\Delta t) =$$

$$= \varepsilon^{-1} \cdot k_1 \cdot \xi_1 \xi_2 \Delta t + O(\Delta t)$$

and

$$P(Y^{(N)}(t+\Delta t) - Y^{(N)}(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mid Y^{(N)}(t) = y) = k_2 y_2 \Delta t + O(\Delta t) =$$

$$= \varepsilon^{-1} \cdot k_2 \cdot \xi_2 \Delta t + O(\Delta t)$$

so that $\alpha(\xi, \{\begin{smallmatrix} -1 \\ +1 \end{smallmatrix}\}) = k_1 \cdot \xi_1 \cdot \xi_2$ and $\alpha(\xi, \{\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\}) = k_2 \xi_2 = k_1 \cdot \rho \cdot \xi_2$, with $\rho = k_2/k_1$.

Let $k_1 = 1$ for convenience.

$$b(\xi) = \begin{pmatrix} -\xi_1 \xi_2 \\ \xi_2(\xi_1 - \rho) \end{pmatrix}$$

$$C(\xi) = \begin{pmatrix} \xi_1 \xi_2 & -\xi_1 \xi_2 \\ -\xi_1 \xi_2 & \xi_2(\xi_1 + \rho) \end{pmatrix}$$

$$B(\xi) = \begin{pmatrix} -\xi_2 & -\xi_1 \\ \xi_2 & \xi_1 - \rho \end{pmatrix}.$$

With $\bar{\xi}'(t) = b(\bar{\xi}(t))$ and noting that $\bar{\xi}_1(t)$ is decreasing two cases occur.

1) $\bar{\xi}_1(0) \leq \rho$ implies that also $\bar{\xi}_2(t)$ is decreasing and no epidemic occurs.

2) $\bar{\xi}_1(0) > \rho$ implies that $\bar{\xi}_2(t)$ is increasing until $\bar{\xi}_1 = \rho$ and an epidemic occurs.

Note that, independently of $\bar{\xi}(0)$, $\lim_{t \to \infty} \bar{\xi}_2(t) = 0$ which is

an absorbing state, where $C(\bar{\xi}) = 0$, so that a compact time interval $[0,T]$ must be studied.

Then $\underline{c}(T) = \inf_{t \leq T} \lambda_1(C(t)) > 0$ if $\bar{\xi}_2(0) > 0$ so that the assumptions of Theorem 7.1 are satisfied under suitable initial assumptions. $\alpha > 0$ in both cases as $B_{22}(\bar{\xi}(0)) = \bar{\xi}_1(0) - \rho > 0$ in case 1) and, with $\tilde{B}_s = -(B+B^T)/2$,

$$\det \tilde{B}_s(\bar{\xi}(t)) = \rho\,\bar{\xi}_2(t) - (\bar{\xi}_1(t)+\bar{\xi}_2(t))^2/4 \to -\bar{\xi}_1^2(\infty)/4 < 0$$

in case 2). Thus, the rate of convergence given by Theorem 7.1 is

$$K(e^{ct} + t^{-1/2})(1 + \underline{c}^{-3}(T))\cdot\varepsilon^{1/2} \quad \text{for} \quad t \leq T.$$

Remark 7.1. Note that these two epidemic models prove that the rate of convergence can not be $K(1 + t^{-1/2})\cdot\varepsilon^{1/2}$ independently of α. If this were the case then, for all $t > 1$ and for all $\varepsilon \leq (\frac{\delta}{2K})^2$ the remainder term would be less than δ which is not possible as with probability 1 the processes get absorbed as $t \to \infty$ giving a remainder term greater than $1/2$.

Remark 7.2. For a more detailed account of these and other epidemic models, see [2].

Example 7.3. Consider a chemical reaction of the type $A + B \rightleftarrows AB$ taking place in a diluted homogeneous solution of volume V.

Let the concentrations of molecules of type A and B be a and b respectively and let the number of AB molecules at time t be $Y(t)$. Then $V\cdot a - Y(t)$ and $V\cdot b - Y(t)$ give the number of molecules of type A and B at time t, so that the situation is completely described by $Y(t)$.

Let $\varepsilon = V^{-1}$ and $\xi^{(\varepsilon)}(t)$ be the concentration of AB molecules at time t, that is $\xi^{(\varepsilon)}(t) = \varepsilon\cdot Y(t)$ and let the infinitesimal transition probabilities be

$$P(Y(t+\Delta t) - Y(t) = +1 \mid Y(t) = y) = (V\cdot a - y)\cdot k_1 \cdot \frac{V\cdot b - y}{V}\Delta t + O(\Delta t) =$$

$$= \varepsilon^{-1}\cdot k_1 \cdot (a - \xi)\cdot(b - \xi)\Delta t + O(\Delta t)$$

and

$$P(Y(t+\Delta t) - Y(t) = -1 \mid Y(t) = y) = k_2 \cdot y \cdot \Delta t + O(\Delta t) =$$

$$= \varepsilon^{-1}\cdot k_2 \cdot \xi \cdot \Delta t + O(\Delta t)$$

so that $\alpha(\xi,\{+1\}) = k_1(a - \xi)\cdot(b - \xi)$ and $\alpha(\xi,\{-1\}) = k_2 \cdot \xi$. This gives

$$b(\xi) = k_1(a - \xi)(b - \xi) - k_2\,\xi,$$

$$B(\xi) = 2k_1\,\xi - (k_1 a + k_1 b + k_2) \leq -k_2$$

as $\xi \leq \min(a,b)$, and

$$C(\xi) = k_1(a - \xi)(b - \xi) + k_2\,\xi.$$

Now

$$R(\xi_1,\xi_2) = (b(\xi_1) - b(\xi_2))/(\xi_1 - \xi_2) \leq \sup_{\xi} B(\xi) \leq -k_2$$

so that $\alpha \leq -k_2 < 0$ and further

$$\underline{c} = \inf_{\xi} C(\xi) > 0 \quad \text{as} \quad \xi \leq \min(a,b).$$

The process in this example is stable, $\alpha < 0$, so that Theorem 7.1 gives the rate of convergence $K(1 + t^{-1/2})\cdot\varepsilon^{1/2}$ under reasonable initial assumptions.

In the above argument the concentrations of molecules of type A and B were assumed constant, a and b respectively, for varying volume V. This is not possible to realize as the number of molecules of a certain type is an integer. Let instead the concentrations be $a^{(\varepsilon)}$ and $b^{(\varepsilon)}$ so that

$$\alpha^{(\varepsilon)}(\xi,\{+1\}) = k_1(a^{(\varepsilon)} - \xi)\cdot(b^{(\varepsilon)} - \xi)$$

and

$$\alpha^{(\varepsilon)}(\xi,\{-1\}) = k_2\,\xi.$$

If $\sup_{\varepsilon} a^{(\varepsilon)} < \infty$, $\sup_{\varepsilon} b^{(\varepsilon)} < \infty$, $|a^{(\varepsilon)} - a| \leq K\cdot\varepsilon$ and $|b^{(\varepsilon)} - b| \leq K\cdot\varepsilon$

then

$$\sup_{\xi} |\alpha^{(\varepsilon)}(\xi,\{+1\}) - \alpha(\xi,\{+1\})| \leq$$
$$\leq k_1 |a^{(\varepsilon)} - a||b^{(\varepsilon)} - \xi| + k_1 |a - \xi| \cdot |b^{(\varepsilon)} - b| \leq K \cdot \varepsilon ,$$

so that (7.1) is satisfied and Theorem 7.2 applies with the same remainder term as above. These conditions are of course satisfied with $a^{(\varepsilon)} = \varepsilon \cdot [\varepsilon^{-1} \cdot a]$ and $b^{(\varepsilon)} = \varepsilon \cdot [\varepsilon^{-1} \cdot b]$, where $|a^{(\varepsilon)} - a| \leq \varepsilon$ and $|b^{(\varepsilon)} - b| \leq \varepsilon$.

Example 7.4. To give an example of a stable multidimensional process consider two chemical reactions taking place simultaneously in a diluted homogeneous solution of volume V:

$$A + B \rightleftarrows AB$$

$$A + C \rightleftarrows AC .$$

Let the concentrations of molecules of type A, B and C be a, b and c respectively (or $a^{(\varepsilon)}$, $b^{(\varepsilon)}$ and $c^{(\varepsilon)}$ as in the preceding example), with $a > b + c$. Let $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, where Y_1 = number of AB-molecules and Y_2 = number of AC-molecules.

With $\varepsilon = V^{-1}$ and $\xi = \varepsilon \cdot Y$ we get

$$\alpha(\xi, A^T) = \begin{cases} k_1 (a - \xi_1 - \xi_2)(b - \xi_1) & \text{for } A = \{(+1,0)\} \\ k_2 (a - \xi_1 - \xi_2)(c - \xi_2) & \text{for } A = \{(0,+1)\} \\ k_3 \xi_1 & \text{for } A = \{(-1,0)\} \\ k_4 \xi_2 & \text{for } A = \{(0,-1)\} \end{cases}$$

with $k_i > 0$, $i = 1,2,3,4$, and

$$b(\xi) = \begin{pmatrix} k_1(a - \xi_1 - \xi_2)(b - \xi_1) - k_3 \xi_1 \\ k_2(a - \xi_1 - \xi_2)(c - \xi_2) - k_4 \xi_2 \end{pmatrix} .$$

Further

$$C(\xi) = \begin{pmatrix} k_1(a - \xi_1 - \xi_2)(b - \xi_1) + k_3 \xi_1 & 0 \\ 0 & k_2(a - \xi_1 - \xi_2)(c - \xi_2) + k_4 \xi_2 \end{pmatrix}$$

so that $\underline{c} = \inf \min (C_{11}(\bar{\xi}(s)), C_{22}(\bar{\xi}(s))) > 0$ as

$$C_{11}(\xi) \geq k_1(c-\xi_2)(b-\xi_1) + k_1(b-\xi_1)^2 + k_3\,\xi_1 > 0$$

and

$$C_{22}(\xi) \geq k_2(c-\xi_2)(b-\xi_1) + k_2(c-\xi_2)^2 + k_4\,\xi_2 > 0.$$

Further

$$B(\xi) = \begin{pmatrix} -k_1(b-\xi_1) - k_1(a-\xi_1-\xi_2) - k_3 & -k_1(b-\xi_1) \\ -k_2(c-\xi_2) & -k_2(c-\xi_2) - k_2(a-\xi_1-\xi_2) - k_4 \end{pmatrix}$$

and the process is stable, $\alpha < 0$, if $\tilde{B} = -(B+B^T)/2$ is strictly positive definite, see (4.3), which is the case if $k_1/8 \leq k_2 \leq 8k_1$ for example. To see this, note that

$$\tilde{B}_{11} \geq k_3 > 0, \quad \tilde{B}_{22} \geq k_4 > 0$$

and

$$\det \tilde{B} \geq (4-\tfrac{1}{2})k_1k_2(b-\xi_1)(c-\xi_2) + (2k_1k_2 - k_1^2/4)(b-\xi_1)^2 +$$

$$+ (2k_1k_2 - k_2^2/4)(c-\xi_2)^2 + k_3k_4 \geq k_3k_4 > 0.$$

Appendix A

This appendix contains the proof of Lemma 4.1 which gives properties of the mean value matrix $\mu(s,t)$ and the covariance matrix $C(s,t)$ of the limit distribution. Some of the properties are well known but are stated and proved for easy reference.

Proof of Lemma 4.1. $\mu(s,t)$ is defined by (2.4). $\frac{\partial}{\partial t}\mu(s,t) = B(t)\mu(s,t)$ with $\mu(s,s) = I$.

(i) The first part of (ii), which does not use (i), gives

$$|\mu(s,t+h) - \mu(s,t)| \leq K(t-s)\cdot h .$$

As $\det \mu(s,s) = \det I = 1$ this implies the existence of a $\delta > 0$ such that if $t - s < \delta$, $\det \mu(s,t) > 1/2$ which in turn implies $\det \mu(s,t) > 0$ for all $s \leq t$. Then $\mu^{-1}(s,t)$ exists for all $s \leq t$ and satisfies

$$\frac{\partial}{\partial t}\mu^{-1}(s,t) = -\mu^{-1}(s,t)B(t) . \tag{A.1}$$

Let $f(s,t) = \mu^{-1}(s,t)\mu(r,t)$. Then $\frac{\partial}{\partial t} f(s,t) = 0$ and $f(s,s) = \mu(r,s)$ that is $\mu(r,s) = \mu^{-1}(s,t)\mu(r,t)$.

(ii) Let $A_i(t)$ be the i:th column of $A(t) = \mu(s,t)$. Then $A'(t) = B(t)A(t)$ so that

$$A_i'(t) = B(t)A_i(t) \quad \text{and} \quad ((A_i(t))^T)' = (A_i(t))^T B^T(t) .$$

Then

$$(|A_i(t)|^2)' = ((A_i(t))^T A_i(t))' = 2(A_i(t))^T((B^T(t) + B(t))/2)A_i(t) ,$$

which gives $-2\bar{a}\,|A_i(t)|^2 \leq (|A_i(t)|^2)' \leq -2\underline{a}\,|A_i(t)|^2$ with $|A_i(s)|^2 = 1$ and thus

$$e^{-2\bar{a}(t-s)} \leq |A_i(t)|^2 \leq e^{-2\underline{a}(t-s)} . \tag{A.2}$$

Now $|\mu(s,t)|^2 = |A(t)|^2 = \sum_i |A_i(s)|^2$ which proves the first part of (ii). The second part is proved by the same method starting from (A.1).

(iii) Let $A_i(t)$ be the i:th column of $\mu(s,t) - I$ and $B_i(t)$ be the i:th column of $B(t)$. Then $A_i'(t) = B(t)A_i(t) + B_i(t)$ and

$$2|A_i(t)|\,|A_i(t)|' = (|A_i(t)|^2)' = ((A_i(t))^T A_i(t))' =$$

$$= 2(A_i(t))^T((B^T(t)+B(t))/2)\,A_i(t) + 2(B_i(t))^T A_i(t) \leq$$

$$\leq -2\underline{a}\,|A_i(t)|^2 + 2\cdot K\cdot|A_i(t)|\,.$$

Dividing by $2|A_i(t)|$ when $t > s$ gives $|A_i(t)|' \leq -\underline{a}|A_i(t)| + K$ with $|A_i(s)| = 0$ so that

$$|A_i(t)| \leq K(1 - e^{-\underline{a}(t-s)})/\underline{a}\,,$$

which gives (iii) as $|\mu(s,t) - I| = (\sum_i |A_i(t)|^2)^{1/2}$.

(iv) Let $f(t) = \det \mu(s,t)$, $f(s) = 1$.

By (i):

$$f'(t) = \lim_{h\to 0}(f(t+h) - f(t))/h = \lim_{h\to 0}(\det \mu(t,t+h) - 1)/h\cdot f(t)$$

and by (iii):

$$|\mu(t,t+h) - I| \leq K\cdot h\cdot e^{|\underline{a}|h}$$

and thus

$$1 - dkh\,e^{|\underline{a}|h} + O(h^2) \leq \det \mu(t,t+h) \leq 1 + dkh\,e^{|\underline{a}|h} + O(h^2)$$

so that $-dk\,f(t) \leq f'(t) \leq dk\,f(t)$ with $f(s) = 1$ which gives the inequality for $\det \mu(s,t)$.

For μ_Ω let $\delta = K\cdot(t-s)e^{|\underline{a}|(t-s)}$ and choose $t - s$ so that $\delta < 1$. Then, by (iii), $\mu_\Omega(s,t) = I + \Delta$ where $|\Delta| \leq \delta$. This gives

$$\det \mu_\Omega(s,t) = \prod_i (1 + \Delta_{ii}) + O(\delta^2) = 1 + O(\delta) > \frac{1}{2}$$

if δ is sufficiently small, that is $t - s \leq \delta_0$.

(v) There exists an x such that

$$\lambda_1(C(s,t))\,|x|^2 = x^T C(s,t)x = \int_s^t x^T\mu(r,t)\,C(r)(x^T\mu(r,t))^T\,dr \geq$$

$$\geq \int_s^t \lambda_1(C(r))\;|x^T\mu(r,t)|^2 dr \geq \underline{c}\;|x|^2 \int_s^t |\mu^{-1}(r,t)|^{-2}\,dr \geq$$

$$> K\cdot|x|^2\,(1 - e^{-2\bar{a}(t-s)})/\bar{a}$$

as $|x| = |x^T| = |x^T \mu \mu^{-1}| \leq |x^T \mu| |\mu^{-1}|$ and by (ii). This gives

$$\lambda_1(C(s,t)) \geq K(1 - e^{-2\bar{a}(t-s)})/\bar{a}$$

and analogously

$$\lambda_d(C(s,t)) \leq K(1 - e^{-2\underline{a}(t-s)})/\underline{a},$$

which gives (v) as $\lambda_1(C(s,t)) \leq |C(s,t)| \leq d^{3/2} \lambda_d(C(s,t))$.

(vi) Follows directly from (v) as $A^T A = C$ implies

$$|C| \leq |A|^2 = \sum_i A_i^T A_i = \sum_i C_{ii} \leq d^{1/2} |C|. \tag{A.3}$$

(vii) By (A.3) it is sufficient to prove

$$K\,\underline{a}(1 - e^{-2\underline{a}(t-s)})^{-1} \leq |C^{-1}(s,t)| \leq K\,\bar{a}(1 - e^{-2\bar{a}(t-s)})^{-1}$$

and this follows from (v) as the eigenvalues of C^{-1} are

$$(\lambda_d(C))^{-1} \leq (\lambda_{d-1}(C))^{-1} \leq \ldots \leq (\lambda_1(C))^{-1}. \tag{A.4}$$

(viii) As $(C^{-1/2}\mu)^T(C^{-1/2}\mu) = \mu^T C^{-1} \mu$, (A.3) gives

$$|\mu^T C^{-1} \mu| \leq |C^{-1/2}\mu|^2 \leq d^{1/2} |\mu^T C^{-1} \mu|. \tag{A.5}$$

Observe that $(\mu^T(s,t)C^{-1}(s,t)\mu(s,t))^{-1} = \mu^{-1}(s,t)C(s,t)(\mu^{-1}(s,t))^T =$
$= \int_s^t \mu^{-1}(s,r)C(r)(\mu^{-1}(s,r))^T dr$ and let this matrix have eigenvalues
$\nu_1 \leq \nu_2 \leq \ldots \leq \nu_d$. Then, by the method in (v), these satisfy

$$\nu_1 \geq K(e^{2\underline{a}(t-s)} - 1)/\underline{a}$$

and

$$\nu_d \leq K(e^{2\bar{a}(t-s)} - 1)/\bar{a},$$

which in combination with (A.4) and (A.5) gives (viii).

(ix) By (A.2): $e^{-2\bar{a}(t-s)} \leq |\mu_i(s,t)|^2 \leq e^{-2\underline{a}(t-s)}$.

$$C_{ii}(s,t) = \int_s^t \sum_{k,\ell} \mu_{ik}(r,t)\, C_{k\ell}(r)\, \mu^T_{\ell i}(r,t) dr \geq$$
$$\geq \underline{c} \int_s^t |\mu_i(r,t)|^2 dr \geq K(1 - e^{-2\bar{a}(t-s)})/\bar{a}.$$

This inequality holds for all i and thus also for $\min_i C_{ii}(s,t)$

Appendix B

In this appendix Lemma 4.2, about the existence of the backward and forward equations of $G^{(\varepsilon)}_{s,t}$, is proved.

The backward equation is

$$-\frac{\partial G^{(\varepsilon)}_{s,t}\varphi}{\partial s} = A^{(\varepsilon)}_s G^{(\varepsilon)}_{s,t}\varphi$$

and the forward equation is

$$\frac{\partial G^{(\varepsilon)}_{s,t}\varphi}{\partial t} = G^{(\varepsilon)}_{s,t} A^{(\varepsilon)}_t \varphi ,$$

where the generator $A^{(\varepsilon)}_s$ has the form

$$(A^{(\varepsilon)}_s\varphi)(x) = \varepsilon^{-1}\int (\varphi(x+\varepsilon^{1/2}z) - \varphi(x))\ \alpha(\bar{\xi}(s)+\varepsilon^{1/2}x,dz) -$$
$$- \varepsilon^{-1/2}(\varphi'(x)\cdot b(\bar{\xi}(s))) .$$

In the backward equation the complication arises that for any $\varphi \in C^1(R^d)$, $G^{(\varepsilon)}_{s,t}\varphi$ must be in $C^1(R^d)$ for $s \le t$. For this reason condition (4.1) in the lemma is introduced.

Proof of Lemma 4.2. $\xi^{(\varepsilon)}(t)$ is a temporally homogeneous jump Markov process with operator $\hat{G}^{(\varepsilon)}_{s,t}$ defined on $C(R^d)$. Let, for $\varphi \in C(R^d)$, $\varphi(\xi,t) = (\hat{G}^{(\varepsilon)}_{0,t}\varphi)(\xi)$. It is well known, see for instance [9], that $\varphi(\xi,t)$ satisfies the backward equation

$$\frac{\partial\varphi(\xi,t)}{\partial t} = \varepsilon^{-1}\int (\varphi(\xi+\varepsilon z,t) - \varphi(\xi,t))\alpha(\xi,dz) = (\hat{A}^{(\varepsilon)}\varphi)(\xi,t) \qquad (B.1)$$

with $\varphi(\xi,0) = \varphi(\xi)$.

But as $\xi^{(\varepsilon)}(t)$ is temporally homogeneous,

$$\hat{G}^{(\varepsilon)}_{s,t} = \hat{G}^{(\varepsilon)}_{0,t-s} \quad \text{and} \quad -\frac{\partial\varphi(\xi,t-s)}{\partial s} = (\hat{A}^{(\varepsilon)}\varphi)(\xi,t-s) . \qquad (B.2)$$

$X^{(\varepsilon)}(t) = \varepsilon^{-1/2}(\xi^{(\varepsilon)}(t) - \bar{\xi}(t))$ where $\bar{\xi}(t)$ is the solution of $\bar{\xi}'(t) = b(\bar{\xi}(t))$.

Let $x = \varepsilon^{-1/2}(\xi - \bar{\xi}(s))$ and $\psi_{s,t}(x) = \varphi(\bar{\xi}(s)+\varepsilon^{1/2}x,t-s)$ so that $\varphi(\xi,t-s) = \psi_{s,t}(\varepsilon^{-1/2}(\xi - \bar{\xi}(s)))$. Then

$$\frac{\partial\psi(\xi,t-s)}{\partial s} = \frac{\partial\psi_{s,t}(x)}{\partial s} + \frac{d\psi_{s,t}(x)}{dx}\cdot\frac{dx}{ds} =$$

$$= \frac{\partial\psi_{s,t}(x)}{\partial s} - \varepsilon^{-1/2}\cdot\left(\frac{d\psi_{s,t}(x)}{dx}\cdot\bar{\xi}'(s)\right).$$

This gives

$$-\frac{\partial\psi_{s,t}(x)}{\partial s} = (\hat{A}^{(\varepsilon)}\varphi)(\xi,t-s) - \varepsilon^{-1/2}\left(\frac{d\psi_{s,t}(x)}{dx}\cdot\bar{\xi}'(s)\right) =$$

$$= \varepsilon^{-1}\int(\varphi(\xi+\varepsilon z,t-s)-\varphi(\xi,t-s))\alpha(\xi,dz) - \varepsilon^{-1/2}\left(\frac{d\psi_{s,t}(x)}{dx}\cdot\bar{\xi}'(s)\right) =$$

$$= \varepsilon^{-1}\int(\psi_{s,t}(x+\varepsilon^{1/2}z)-\psi_{s,t}(x))\alpha(\bar{\xi}(s)+\varepsilon^{1/2}x,dz) - \varepsilon^{-1/2}\left(\frac{d\psi_{s,t}(x)}{dx}\cdot\bar{\xi}'(s)\right) =$$

$$= (A_s^{(\varepsilon)}\psi_{s,t})(x),$$

with $\psi_{t,t}(x) = \varphi(\bar{\xi}(t)+\varepsilon^{1/2}x,0) = \varphi(\bar{\xi}(t)+\varepsilon^{1/2}x)$.

Define $\psi(x) = \varphi(\bar{\xi}(t)+\varepsilon^{1/2}x)$ for $\varphi \in C(R^d)$. The class of all such ψ equals $C(R^d)$. Then

$$(G_{s,t}^{(\varepsilon)}\psi)(x) = E(\psi(X^{(\varepsilon)}(t)) \mid X^{(\varepsilon)}(s)=x) =$$

$$= E(\varphi(\xi^{(\varepsilon)}(t)) \mid \xi^{(\varepsilon)}(s)=\xi) = \varphi(\xi,t-s)$$

which satisfies (B.2). This implies that $G_{s,t}^{(\varepsilon)}\psi$ satisfies the backward equation

$$-\frac{\partial(G_{s,t}^{(\varepsilon)}\psi)(x)}{\partial s} = (A_s^{(\varepsilon)}G_{s,t}^{(\varepsilon)}\psi)(x) \quad \text{with} \quad (G_{t,t}^{(\varepsilon)}\psi)(x) = \psi(x)$$

for all $\psi \in C(R^d)$ for which the equation is meaningful, that is for which $\frac{\partial(G_{s,t}^{(\varepsilon)}\psi)(x)}{\partial x_k}$ exists finite for all k, x and $s \le t$. As $G_{t,t}^{(\varepsilon)}\psi = \psi$, this implies that $\psi \in C^1(R^d)$. But

$$\frac{\partial}{\partial x_k}(G_{s,t}^{(\varepsilon)}\psi)(x) = \frac{\partial}{\partial x_k}\psi_{s,t}(x) = \frac{\partial}{\partial x_k}\varphi(\bar{\xi}(s)+\varepsilon^{1/2}x,t-s)$$

so that $\frac{\partial}{\partial x_k}(G_{s,t}^{(\varepsilon)}\psi)(x)$ exists if and only if $\frac{\partial}{\partial x_k}\varphi(x,t) = \varphi_k'(x,t)$ exists. By (B.1),

$$\frac{\partial}{\partial t}\varphi(x,t) = (\hat{A}^{(\varepsilon)}\varphi)(x,t) = \varepsilon^{-1}\int(\varphi(x+\varepsilon z,t)-\varphi(x,t))\alpha(x,dz).$$

For simplicity of notations replace $\hat{A}^{(\varepsilon)}$ by A and $\hat{G}_{0,t}^{(\varepsilon)}$ by G_t.

Let $A'_k = \frac{\partial A}{\partial x_k}$ be defined by

$$(A'_k\varphi)(x) = \varepsilon^{-1} \int (\varphi(x+\varepsilon z) - \varphi(x))\ \alpha'_k(x,dz)$$

which is well defined by condition (4.1). Observe now that if $\varphi'_k(x,t)$ exists, it will, according to (B.1), formally be given by

$$\varphi'_k(x,t) = \int_0^t (G_{t-s}(A'_k\varphi))(x,s)ds + (G_t\varphi'_k)(x) \qquad (B.3)$$

as $\varphi'_k(x,t) - (G_t\varphi'_k)(x) = \int_0^t \frac{\partial}{\partial s}(G_{t-s}\varphi'_k)(x,s)ds = \int_0^t G_{t-s}(A'_k\varphi)(x,s)ds$. But $G_t = e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$ and thus

$$\int_0^t G_{t-s}(A'_k\varphi)(x,s)ds + (G_t\varphi'_k)(x) =$$

$$= \int_0^t \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (\frac{A^i(t-s)^i}{i!} A'_k \frac{A^j s^j}{j!}\varphi)(x)ds + (G_t\varphi'_k)(x) =$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{n} (\frac{A^{n-m}A'_kA^{m-1}}{(n-m)!(m-1)!}\varphi)(x) \int_0^t (t-s)^{n-m}s^{m-1}ds + (G_t\varphi'_k)(x) =$$

$$= \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{m=1}^{n} (A^{n-m}A'_k A^{m-1}\varphi)(x) + \sum_{n=0}^{\infty} (\frac{A^n t^n}{n!}\varphi'_k)(x) =$$

$$= \sum_{n=1}^{\infty} \frac{t^n}{n!} (\frac{\partial}{\partial x_k}(A^n\varphi)(x) - (A^n\varphi'_k)(x)) + \sum_{n=0}^{\infty} \frac{t^n}{n!}(A^n\varphi'_k)(x) =$$

$$= \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{\partial}{\partial x_k}(A^n\varphi)(x) + \varphi'_k(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial}{\partial x_k}(A^n\varphi)(x).$$

Then

$$\int_0^y \{\int_0^t G_{t-s}(A'_k\varphi)(x,s)ds + (G_t\varphi'_k)(x)\}dx_k =$$

$$= \int_0^y \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial}{\partial x_k}(A^n\varphi)(x)dx_k = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^y \frac{\partial}{\partial x_k}(A^n\varphi)(x)dx_k =$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!}(A^n\varphi)(y) + C = (G_t\varphi)(y) + C.$$

Thus $(G_t\varphi)(x)$ is differentiable with respect to x_k with derivative given by (B.3).

It remains to be shown that the derivative is bounded for fixed t but

$$\sup_x |\varphi_k'(x,t)| \le t \cdot \sup_x \sup_{0\le s\le t} |(A_k'\varphi)(x,s)| + \sup_x |\varphi_k'(x)| \le$$

$$\le 2 \cdot t \cdot \varepsilon^{-1} \sup_x |\varphi(x)| \cdot \sup_x \int |\alpha_k'(x,dz)| + \sup_x |\varphi_k'(x)|$$

which is bounded for $\varphi \in C^1(R^d)$ as (4.1) holds.

To show that the forward equation is satisfied, observe that for $h > 0$

$$h^{-1}(G^{(\varepsilon)}_{s,t+h} - G^{(\varepsilon)}_{s,t}) = G^{(\varepsilon)}_{s,t}\, h^{-1}(G^{(\varepsilon)}_{t,t+h} - I) .$$

But $\lim_{h\to 0} h^{-1}(G^{(\varepsilon)}_{t,t+h} - I) = A^{(\varepsilon)}_t$ by the above as

$$\lim_{h\to 0} h^{-1}(G^{(\varepsilon)}_{s+h,t} - G^{(\varepsilon)}_{s,t}) = \lim_{h\to 0} h^{-1}(I - G^{(\varepsilon)}_{s,s+h})\, G^{(\varepsilon)}_{s+h,t} = -A^{(\varepsilon)}_s\, G^{(\varepsilon)}_{s,t}$$

and

$$\lim_{h\to 0} G^{(\varepsilon)}_{s+h,t} = G^{(\varepsilon)}_{s,t} .$$

On the other hand

$$\lim_{h\to 0} h^{-1}(G^{(\varepsilon)}_{s,t} - G^{(\varepsilon)}_{s,t-h}) = \lim_{h\to 0} G^{(\varepsilon)}_{s,t-h} \cdot h^{-1}(G^{(\varepsilon)}_{t-h,t} - I) = \lim_{h\to 0} G^{(\varepsilon)}_{s,t-h} \cdot A^{(\varepsilon)}_t$$

and

$$\lim_{h\to 0} |(G^{(\varepsilon)}_{s,t} - G^{(\varepsilon)}_{s,t-h})\varphi(x)| = \lim_{h\to 0} |(\hat{G}^{(\varepsilon)}_{t-s} - \hat{G}^{(\varepsilon)}_{t-s-h})\psi(\bar{\xi}(s) + \varepsilon^{1/2}x)| \le$$

$$\le \lim_{h\to 0} \Big| \int_0^h (\hat{A}^{(\varepsilon)} \hat{G}^{(\varepsilon)}_{t-s-r} \psi)(\bar{\xi}(s) + \varepsilon^{1/2}x)\, dr \Big| = 0 .$$

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